

Minimal surfaces with two ends which have the least total absolute curvature

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Abstract

In this paper, we consider complete non-catenoidal minimal surfaces of finite total curvature with two ends. A family of such minimal surfaces with least total absolute curvature is given. Moreover, we obtain a uniqueness theorem for this family from its symmetries.

1 Introduction

For a complete minimal surface in Euclidean space, an inequality stronger than the classical inequality of Cohn-Vossen holds, giving a lower bound for the total absolute curvature. It is then natural to ask whether there is a minimal surface which attains this minimum value for the total absolute curvature. We consider this problem and contribute to the theory of existence of minimal surfaces in Euclidean space. Our work connects with the Björling problem for minimal surfaces in Euclidean space.

Let $f : M \rightarrow \mathbb{R}^3$ be a minimal immersion of a 2-manifold M into Euclidean 3-space \mathbb{R}^3 , and we usually call f a *minimal surface* in \mathbb{R}^3 . Choosing isothermal coordinates makes M a Riemann surface, and then f is called a *conformal minimal immersion*. The following representation formula is one of the basic tools in the theory of minimal surfaces:

Theorem 1.1 (Weierstrass representation [O]). *Let (g, η) be a pair of a meromorphic function g and a holomorphic differential η on a Riemann surface M so that*

$$(1 + |g|^2)^2 \eta \bar{\eta} \quad (1.1)$$

gives a Riemannian metric on M . We set

$$\Phi := \begin{pmatrix} (1 - g^2)\eta \\ i(1 + g^2)\eta \\ 2g\eta \end{pmatrix}, \quad (1.2)$$

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where $i = \sqrt{-1}$. Assume that

$$\operatorname{Re} \oint_{\ell} \Phi = \mathbf{0} \quad \text{holds for any } \ell \in \pi_1(M). \quad (\text{P})$$

Then

$$f = \operatorname{Re} \int_{z_0}^z \Phi : M \rightarrow \mathbb{R}^3 \quad (z_0 \in M) \quad (1.3)$$

defines a conformal minimal immersion.

The pair (g, η) in Theorem 1.1 is called the *Weierstrass data* of f .

Remark 1.2. Condition (P) is called the *period condition* of the minimal surface. (P) is equivalent to

$$\oint_{\ell} \eta = \overline{\oint_{\ell} g^2 \eta} \quad (1.4)$$

and

$$\operatorname{Re} \oint_{\ell} g \eta = 0 \quad (1.5)$$

for any $\ell \in \pi_1(M)$.

Remark 1.3. The first fundamental form ds^2 and the second fundamental form \mathbb{I} of the surface (1.3) are given by

$$ds^2 = (1 + |g|^2)^2 \eta \bar{\eta}, \quad \mathbb{I} = -\eta dg - \overline{\eta} d\bar{g}.$$

Moreover, $g : M \rightarrow \mathbb{C} \cup \{\infty\}$ coincides with the composition of the Gauss map $G : M \rightarrow S^2$ of the minimal surface and stereographic projection $\sigma : S^2 \rightarrow \mathbb{C} \cup \{\infty\}$, that is, $g = \sigma \circ G$. So we call g the Gauss map of the minimal surface.

Next, we assume that a minimal surface is complete and of finite total curvature. These two conditions give rise to restrictions on the topological and conformal types of minimal surfaces.

Theorem 1.4 ([H, O]). *Let $f : M \rightarrow \mathbb{R}^3$ be a conformal minimal immersion. Suppose that f is complete and of finite total curvature. Then, the following hold:*

- (1) *M is conformally equivalent to a compact Riemann surface \overline{M}_{γ} of genus γ punctured at a finite number of points p_1, \dots, p_n .*
- (2) *The Gauss map g extends to a holomorphic mapping $\hat{g} : \overline{M}_{\gamma} \rightarrow \mathbb{C} \cup \{\infty\}$.*

Removed points p_1, \dots, p_n correspond to ends of the minimal surface.

The asymptotic behavior around each end p_i can be described by the order of the poles of $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ in Theorem 1.1 at p_i . Let

$$d_i = \max_{1 \leq j \leq 3} \{\operatorname{ord}(\Phi_j, p_i)\} - 1, \quad (1.6)$$

where $\text{ord}(\Phi_j, p_i)$ is the order of the pole of Φ_j at p_i ($1 \leq i \leq n$, $1 \leq j \leq 3$). Condition (P) yields $\text{residue}(\Phi, p_i) \in \mathbb{R}^3$, and thus $d_i \geq 1$. The following theorem shows the geometric properties of d_i , which includes a stronger inequality than the Cohn-Vossen inequality:

Theorem 1.5 ([O, JM, Sc]). *Let $f : M \rightarrow \mathbb{R}^3$ be a minimal surface as in Theorem 1.4.*

(a) *The immersion f is proper.*

(b) *If $S^2(r)$ is the sphere of radius r , then $\frac{1}{r}(f(M) \cap S^2(r))$ consists of n closed curves $\Gamma_1, \dots, \Gamma_n$ in $S^2(1)$ which converge C^1 to closed geodesics $\gamma_1, \dots, \gamma_n$ of $S^2(1)$, with multiplicities d_1, \dots, d_n , as $r \rightarrow \infty$. Moreover,*

$$\frac{1}{2\pi} \int_M K dA = \chi(\overline{M}_\gamma) - \sum_{i=1}^n (d_i + 1) \leq \chi(M) - n = \chi(\overline{M}_\gamma) - 2n = 2(1 - \gamma - n), \quad (1.7)$$

and equality holds if and only if each end is embedded.

The equation on the left of (1.7) is called the *Jorge-Meeks formula*.

Moreover, a relation between the total (absolute) curvature and the degree of g is as follows. Note that since g extends to a holomorphic map \hat{g} from a compact Riemann surface \overline{M}_γ to a compact Riemann surface $\mathbb{C} \cup \{\infty\}$, we can define the degree of g by $\deg(g) := \deg(\hat{g})$. Since the Gaussian curvature of a minimal surface $M \rightarrow \mathbb{R}^3$ is always non-positive, its total absolute curvature $\tau(M) := \int_M |K| dA$ is given by

$$\tau(M) = \int_M (-K) dA.$$

Recall that the total absolute curvature of a minimal surface in \mathbb{R}^3 is just the area under the Gauss map $g : M \rightarrow \mathbb{C} \cup \{\infty\} \cong S^2$, that is,

$$\tau(M) = (\text{the area of } S^2) \deg(g) = 4\pi \deg(g) \in 4\pi \mathbb{Z}.$$

(See, for example, (3.11) in [HO] for details.) Hence (1.7) is rewritten as

$$\deg(g) \geq \gamma + n - 1 \quad (1.8)$$

and we consider sharpness of the inequality (1.8).

For $n \geq 3$, there exist many examples of minimal surfaces which satisfy $\deg(g) = \gamma + n - 1$. (See Figure 1.1.)

If $n = 1$, then a minimal surface satisfying $\deg(g) = \gamma$ must be a plane. (See, for instance, [HK, Remark 2.2].) Thus on a non-planar minimal surface, $\deg(g) \geq \gamma + 1$. The existence of minimal surfaces with $\deg(g) = \gamma + 1$ was shown by C. C. Chen and F. Gackstatter [CG] (for $\gamma = 1, 2$), N. Do Espirito-Santo [ES] (for $\gamma = 3$), K. Sato [Sa], and M. Weber and M. Wolf [WW]. (See Figure 1.2.)

Finally, we consider the case $n = 2$. In this case, the following uniqueness result is known:

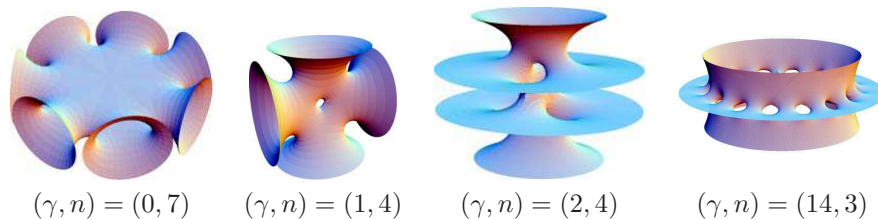


Figure 1.1: Minimal surfaces with $n \geq 3$ satisfying equality in (1.8). For details on these surfaces, see, for instance, [JM], [BR], [Wo], [HM].

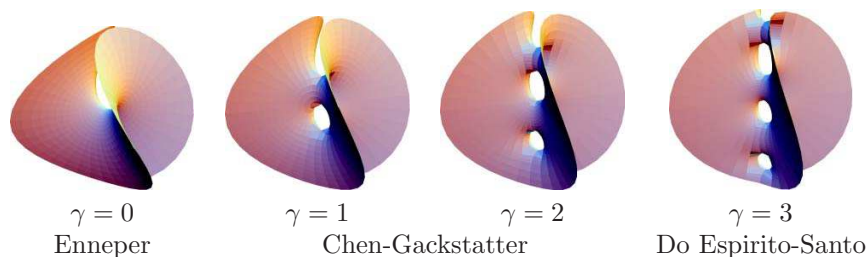


Figure 1.2: Minimal surfaces with $n = 1$ satisfying $\deg(g) = \gamma + 1$.

Theorem 1.6 ([Sc]). *Let $f : M \rightarrow \mathbb{R}^3$ be a complete conformal minimal surface of finite total curvature. If f has two ends and equality holds in (1.8), then f must be a catenoid.*

It follows that on a non-catenoidal minimal surface with two ends,

$$\deg(g) \geq \gamma + 2. \quad (1.9)$$

As a consequence, it is reasonable to consider the following problem:

Problem 1.7. For an arbitrary genus γ , does there exist a complete conformal minimal surface of finite total curvature with two ends which satisfies equality in (1.9)?

In the case $\gamma = 0$ such minimal surfaces exist, and moreover, these minimal surfaces have been classified by F. J. López [L1]. (See Figure 1.3.)

However, for the case $\gamma > 0$ no answer to Problem 1.7 is known. Our first main result is to give a partial answer to this problem:

Main Theorem 1. *If γ is equal to 1 or an even number, there exists a complete conformal minimal surface of finite total curvature with two ends which satisfies equality in (1.9).*

Note that if we do not assume the equality in (1.9), then there exists a complete conformal minimal surface of finite total curvature with two ends for an arbitrary genus $\gamma \geq 0$. (See, for instance, [FS].)

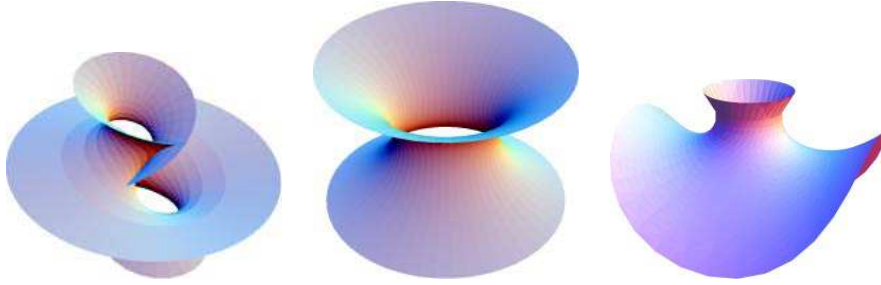


Figure 1.3: Examples for $\gamma = 0$. The surface in the middle is a double cover of a catenoid.

We shall prove Main Theorem 1 by explicit constructions. (See §2.) We now discuss the asymptotic behavior for our minimal surfaces in terms of d_i . For a minimal surface as in Problem 1.7, we have $(d_1, d_2) = (1, 3), (2, 2)$. The case $(d_1, d_2) = (1, 3)$ corresponds to a minimal surface with an embedded end and an Enneper's type end. Recall that an embedded end is asymptotic to a plane or a catenoid. (See [Sc].) The minimal surface given in §2.1 has an embedded end which is asymptotic to a plane and $(d_1, d_2) = (1, 3)$. (see Corollary 2.3.) The minimal surface introduced in §4.1 is another example with an embedded end which is asymptotic to a half catenoid and $(d_1, d_2) = (1, 3)$. The minimal surfaces with $(d_1, d_2) = (2, 2)$ are obtained in §2.2. (See Corollary 2.6.)

The minimal surfaces given in Corollary 2.3 and Corollary 2.6 have symmetry groups with $4(\gamma + 1)$ elements. Next we consider the uniqueness theorem for the symmetries. Uniqueness is also one of the important problems for minimal surfaces, and there are many uniqueness theorems. (See [MaW, HM].) Our other main theorem is as follows.

Main Theorem 2. *Let $f : M \rightarrow \mathbb{R}^3$ be a complete conformal minimal surface of finite total curvature with two ends and genus γ . Suppose that f satisfies equality in (1.9) and has $4(\gamma + 1)$ symmetries. We assume either $\gamma = 1$ and $(d_1, d_2) = (1, 3)$, or γ is an even number and $(d_1, d_2) = (2, 2)$. Then f is one of the minimal surfaces given in Main Theorem 1.*

At the end of this section, we discuss our work from the point of view of the Björling problem for minimal surfaces. The classical Björling problem is to determine a piece of a minimal surface containing a given analytic strip. This was named after E. G. Björling in 1844. H. A. Schwarz gave an explicit solution to it. (See, for instance, [N].) Recently, Mira [Mi] used the solution to the Björling problem to classify a certain class of minimal surfaces of genus 1. Also, Meeks and Weber [MW] produced an infinite sequence of complete minimal annuli by using the solution to the Björling problem and then gave a complete answer as to which curves appear as the singular set of a Colding-Minicozzi limit minimal lamination. Hence it is useful to study minimal surfaces from the point of view of the Björling problem. However, the existence of minimal surfaces

of higher genus derived from the solution to the Björling problem seems to be unknown. In Section 2.2, we show that our minimal surfaces, which have even numbers for the genus, are solutions to the Björling problem, and the generating curves are closed plane curves.

The paper is organized as follows: Section 2 contains constructions of concrete examples to prove Main Theorem 1 and is divided into two subsections. The genus 1 case is provided in Section 2.1 and even number cases are provided in Section 2.2. Section 2.2 also contains the result from the point of view of the Björling problem. Moreover, we prove our uniqueness result in Section 3. In Section 4 we refer to remaining problems related to our work.

2 Construction of surfaces for Main Theorem 1

In this section we will construct the surfaces for proving Main Theorem 1 in the introduction. We will use the Weierstrass representation in Theorem 1.1, for which we need a Riemann surface M , a meromorphic function g , and a holomorphic differential η .

2.1 The case $\gamma = 1$

Let \overline{M}_γ be the Riemann surface

$$\overline{M}_\gamma = \{(z, w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^{\gamma+1} = z(z^2 - 1)^\gamma\}.$$

The surface we will consider is

$$M = \overline{M}_\gamma \setminus \{(0, 0), (\infty, \infty)\},$$

a Riemann surface of genus γ from which two points have been removed. We want to define a complete conformal minimal immersion of M into \mathbb{R}^3 by the Weierstrass representation in Theorem 1.1. To do this, set

$$g = cw, \quad \eta = i \frac{dz}{z^2 w},$$

where $c \in \mathbb{R}_{>0}$ is a positive constant to be determined.

Let Φ be the \mathbb{C}^3 -valued differential as in (1.2). We shall prove that (1.3) is a conformal minimal immersion of M .

We begin with, we show by straightforward calculation how the following conformal diffeomorphisms κ_1 and κ_2 act on Φ .

Lemma 2.1 (Symmetries of the surface). *Consider the following conformal mappings of M :*

$$\kappa_1(z, w) = (\bar{z}, \bar{w}), \quad \kappa_2(z, w) = (-z, e^{\pi i/(\gamma+1)} w).$$

Then,

$$\kappa_1^* \Phi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \bar{\Phi}, \quad \kappa_2^* \Phi = \begin{pmatrix} -\cos \frac{\pi}{\gamma+1} & \sin \frac{\pi}{\gamma+1} & 0 \\ -\sin \frac{\pi}{\gamma+1} & -\cos \frac{\pi}{\gamma+1} & 0 \\ 0 & 0 & -1 \end{pmatrix} \Phi.$$

Since (1.1) gives a complete Riemannian metric on M (see Table 2.1), it suffices to show that f is well-defined on M for the right choice of c .

(z, w)	$(0, 0)$	$(1, 0)$	$(-1, 0)$	(∞, ∞)
g	0^1	0^γ	0^γ	$\infty^{2\gamma+1}$
η	$\infty^{\gamma+3}$			$0^{3\gamma+1}$

Table 2.1: Orders of zeros and poles of g and η .

Theorem 2.2. *For any positive number γ , there exists a unique positive constant $c \in \mathbb{R}_{>0}$ for which the immersion f given in (1.3) is well-defined on M .*

Proof. To establish this theorem we must show (P) in Theorem 1.1. We will prove (1.4) and (1.5), respectively. (1.5) follows from the exactness of $g\eta = icdz/z^2 = d(-ic/z)$, and thus we will only have to show (1.4). We first check the residues of η and $g^2\eta$ at the ends $(0, 0)$, (∞, ∞) . At the end $(0, 0)$, w is a local coordinate for the Riemann surface \overline{M}_γ , and then $z = z(w) = w^{\gamma+1}\{(-1)^\gamma + \mathcal{O}(w^{2\gamma+2})\}$. We have

$$\eta = \left(\frac{\alpha_1}{w^{\gamma+3}} + \mathcal{O}(w^{\gamma-1}) \right) dw \quad \text{and} \quad g^2\eta = \left(\frac{\alpha_2}{w^{\gamma+1}} + \mathcal{O}(w^{\gamma+1}) \right) dw,$$

where $\alpha_j \in \mathbb{C}$ ($j = 1, 2$) are constants. These imply that both η and $g^2\eta$ have no residues at $(0, 0)$. Then the residue theorem yields that they have no residues at (∞, ∞) as well.

We next consider path-integrals along topological 1-cycles on \overline{M}_γ . We will give a convenient 1-cycle.

Define a 1-cycle on \overline{M}_γ as

$$\begin{aligned} \ell = & \left\{ (z, w) = \left(-t, \sqrt[\gamma+1]{-t(1-t^2)^\gamma} e^{\gamma\pi i/(\gamma+1)} \right) \mid -1 \leq t \leq 0 \right\} \\ & \cup \left\{ (z, w) = \left(t, \sqrt[\gamma+1]{t(1-t^2)^\gamma} e^{-\gamma\pi i/(\gamma+1)} \right) \mid 0 \leq t \leq 1 \right\}. \end{aligned}$$

Recall that $(0, 0)$ corresponds to the end of f . Avoiding the end $(0, 0)$, we can deform ℓ to a 1-cycle ℓ' on M which is projected to a loop winding once around $[0, 1]$ in the z -plane. (See Figure 2.1.)

By the actions of the κ_j 's, we can obtain all of the 1-cycles on M from ℓ' . If (P) holds for this ℓ' , then

$$\operatorname{Re} \int_{\kappa_j \circ \ell'} \Phi = \operatorname{Re} \int_{\ell'} \kappa_j^* \Phi = K \operatorname{Re} \int_{\ell'} \Phi = \mathbf{0}$$

for some orthogonal matrix K , by Lemma 2.1. Hence all that remains to be done is to show that (1.4) holds for ℓ' .

We now calculate path-integrals of η and $g^2\eta$ along ℓ' , and we want to reduce them to path-integrals along ℓ for simplicity. Note that both η and $g^2\eta$ have

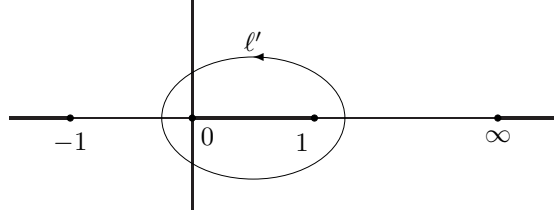


Figure 2.1: Projection to the z -plane of the loop $\ell' \in \pi_1(M)$.

poles at $(0, 0)$. To avoid divergent integrals, here we add exact 1-forms which have principal parts of η and $g^2\eta$, respectively. It is straightforward to check

$$\frac{dz}{z^2w} - \frac{\gamma+1}{\gamma+2}d\left(\frac{z^2-1}{zw}\right) = \frac{\gamma}{\gamma+2}\frac{dz}{w}, \quad \frac{w}{z^2}dz + \frac{\gamma+1}{\gamma}d\left(\frac{w}{z}\right) = \frac{2w}{z^2-1}dz.$$

So we have

$$\oint_{\ell'} \eta = \frac{i\gamma}{\gamma+2} \oint_{\ell'} \frac{dz}{w} = \frac{i\gamma}{\gamma+2} \oint_{\ell} \frac{dz}{w} = \frac{-2\gamma}{\gamma+2} \sin \frac{\gamma\pi}{\gamma+1} \int_0^1 \frac{dt}{\gamma^{+1}\sqrt{t(1-t^2)}^\gamma},$$

$$\oint_{\ell'} g^2\eta = 2ic^2 \oint_{\ell'} \frac{w}{z^2-1}dz = 2ic^2 \oint_{\ell} \frac{w}{z^2-1}dz = -4c^2 \sin \frac{\gamma\pi}{\gamma+1} \int_0^1 \gamma^{+1}\sqrt{\frac{t}{1-t^2}}dt.$$

By setting

$$A_\gamma = \frac{\gamma}{\gamma+2} \int_0^1 \frac{dt}{\gamma^{+1}\sqrt{t(1-t^2)}^\gamma} \in \mathbb{R}_{>0}, \quad B_\gamma = 2 \int_0^1 \gamma^{+1}\sqrt{\frac{t}{1-t^2}}dt \in \mathbb{R}_{>0},$$

(1.4) is reduced to $A_\gamma = c^2 B_\gamma$. Let us set

$$c = \sqrt{\frac{A_\gamma}{B_\gamma}} \in \mathbb{R}_{>0}.$$

This choice of c satisfies (1.4) and is the unique positive real number that does so. This completes the proof. (See Figure 2.2.) \square

Since $\deg(g) = 2\gamma+1$, $\deg(g) = \gamma+2$ if and only if $\gamma = 1$. As a consequence, the next corollary follows:

Corollary 2.3. *There exists a complete conformal minimal surface of genus 1 with two ends which has least total absolute curvature.*

2.2 The case γ is even

The following construction is similar to the construction in Section 2.1. Crucial arguments are given after (2.3).

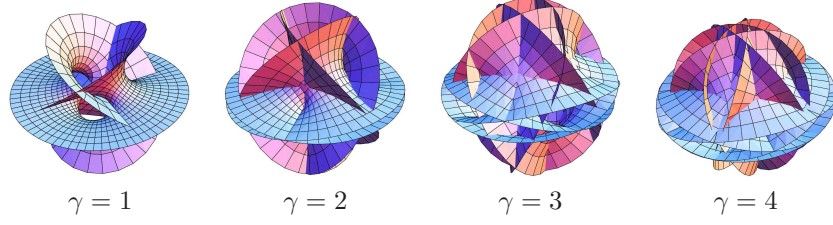


Figure 2.2: Minimal surfaces of genus γ with two ends which satisfy $\deg(g) = 2\gamma + 1$.

For an integer $k \geq 2$, let \overline{M}_γ be the Riemann surface

$$\overline{M}_\gamma = \left\{ (z, w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^{k+1} = z^2 \left(\frac{z-1}{z-a} \right)^k \right\},$$

where $a \in (1, \infty)$ is a constant to be determined. By the Riemann-Hurwitz formula, we see that

$$\gamma = \begin{cases} k & (\text{if } k \text{ is even}), \\ k-1 & (\text{if } k \text{ is odd}). \end{cases}$$

Note that the genus γ is always even. (See Figure 2.3.)

We set

$$M = \overline{M}_\gamma \setminus \{(0, 0), (\infty, \infty)\}, \quad g = cw \quad \left(c = a^{(k-2)/(2k+2)} \in \mathbb{R}_{>0} \right), \quad \eta = \frac{dz}{zw}.$$

Then (1.1) gives a complete Riemannian metric M . (See Table 2.2.)

(z, w)	$(0, 0)$	$(1, 0)$	(a, ∞)	(∞, ∞)
g	0^2	0^k	∞^k	∞^2
η	∞^4		0^{2k}	

(z, w)	$(0, 0)$	$(1, 0)$	(a, ∞)	(∞, ∞)
g	0^2	0^k	∞^k	∞^2
η	∞^3		0^{2k}	0^1

Table 2.2: Orders of zeros and poles of g and η when k is odd (top) and k is even (bottom).

Also, $\deg(g) = k + 2$ for all $k \geq 2$. Thus equality in (1.9) holds if and only if k is even. Hereafter we assume k is even.

Let Φ be the \mathbb{C}^3 -valued differential as in (1.2). We shall prove that (1.3) is a conformal minimal immersion of M .

First, we observe the following symmetries κ_1 , κ_2 , and κ_3 of the surface.

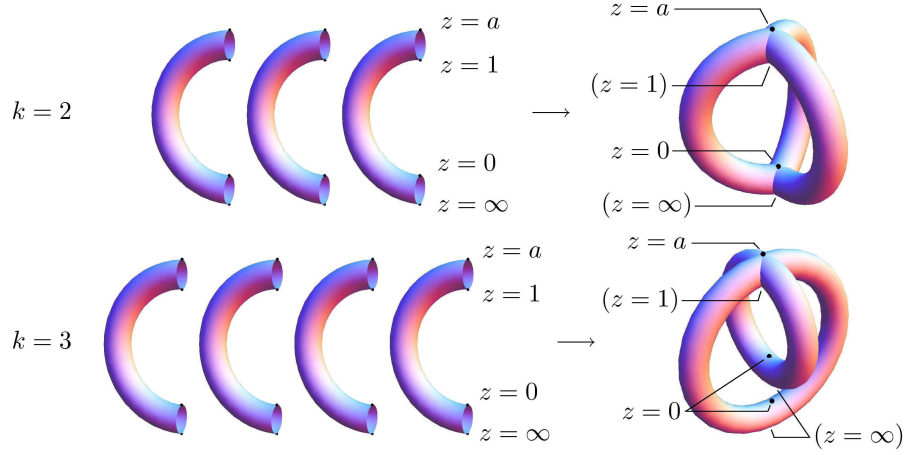


Figure 2.3: Riemann surfaces \overline{M}_γ , $k = 2$ (top) and $k = 3$ (bottom). Both surfaces have genus 2. When k is odd, \overline{M}_γ has self-intersections near $z = 0$ and $z = \infty$. In the sketch in the bottom row, we see two different $z = 0$ (and two different $z = \infty$, which are hidden from this viewpoint) but they are in fact the same points. The reason we place these points differently is to reveal their genus clearly.

Lemma 2.4 (Symmetries of the surface). *Consider the following conformal mappings of M :*

$$\begin{aligned}\kappa_1(z, w) &= (\bar{z}, \bar{w}), & \kappa_2(z, w) &= (z, e^{2\pi i/(k+1)} w), \\ \kappa_3(z, w) &= \left(\frac{a}{z}, \frac{1}{c^2 w} \right).\end{aligned}$$

Then,

$$\begin{aligned}\kappa_1^* \Phi &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \bar{\Phi}, & \kappa_2^* \Phi &= \begin{pmatrix} \cos \frac{2\pi}{k+1} & -\sin \frac{2\pi}{k+1} & 0 \\ \sin \frac{2\pi}{k+1} & \cos \frac{2\pi}{k+1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \Phi, \\ \kappa_3^* \Phi &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Phi.\end{aligned}$$

As we have already seen the completeness of f , it suffices to show that f is well-defined on M for the right choice of $a \in (1, \infty)$.

Theorem 2.5. *For any positive even number k , there exists a unique constant $a \in (1, \infty)$ for which the immersion f given in (1.3) is well-defined on M .*

Proof. We will show (P) in Theorem 1.1. It is easy to verify that there are no residues at the ends $(0, 0)$, (∞, ∞) . So all that remains is to choose c so that

(P) is satisfied. (1.5) follows from the exactness of $g\eta = (c/z)dz = c \cdot d(\log z)$ and $c \in \mathbb{R}$, and hence we will only have to show (1.4). To do this, we will give convenient 1-cycles.

Define a 1-cycle on \overline{M}_γ as

$$\begin{aligned} \ell_1 = & \left\{ (z, w) = \left(-t, \sqrt[k+1]{t^2 \left(\frac{1+t}{a+t} \right)^k} \right) \mid -1 \leq t \leq 0 \right\} \\ & \cup \left\{ (z, w) = \left(t, \sqrt[k+1]{t^2 \left(\frac{1-t}{a-t} \right)^k} e^{2\pi i/(k+1)} \right) \mid 0 \leq t \leq 1 \right\}. \end{aligned}$$

Recall that $(0,0)$ corresponds to the end of f . Avoiding the end $(0,0)$, we can deform ℓ_1 to a 1-cycle ℓ'_1 on M which is projected to a loop winding once around $[0, 1]$ in the z -plane. We also define another 1-cycle on M as

$$\begin{aligned} \ell_2 = & \left\{ (z, w) = \left(-t, \sqrt[k+1]{t^2 \left(\frac{-t-1}{a+t} \right)^k} e^{k\pi i/(k+1)} \right) \mid -a \leq t \leq -1 \right\} \\ & \cup \left\{ (z, w) = \left(t, \sqrt[k+1]{t^2 \left(\frac{t-1}{a-t} \right)^k} e^{-k\pi i/(k+1)} \right) \mid 1 \leq t \leq a \right\}. \end{aligned}$$

(See Figure 2.4.)

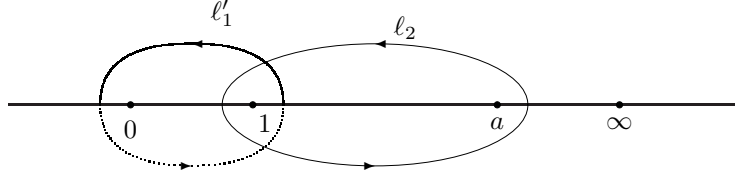


Figure 2.4: Projections to the z -plane of the loops ℓ'_1 and $\ell_2 \in \pi_1(M)$.

Again by the actions of the κ_j 's, we can obtain all of the 1-cycles on M from ℓ'_1 and ℓ_2 . We now show that (1.4) holds for ℓ'_1 and ℓ_2 .

First we calculate the path-integrals of η and $g^2\eta$ along ℓ_2 . Then we have

$$\oint_{\ell_2} \eta = 2i \sin \frac{k\pi}{k+1} \int_1^a \sqrt[k+1]{\frac{(a-t)^k}{t^{k+3}(t-1)^k}} dt, \quad (2.1)$$

$$\begin{aligned} \oint_{\ell_2} g^2\eta &= -2ic^2 \sin \frac{k\pi}{k+1} \int_1^a \sqrt[k+1]{\frac{(t-1)^k}{t^{k+3}(a-t)^k}} dt \\ &= -2i \sin \frac{k\pi}{k+1} \int_1^a \sqrt[k+1]{\frac{(a-\tau)^k}{\tau^{k+3}(\tau-1)^k}} d\tau, \end{aligned} \quad (2.2)$$

where $\tau = a/t$. As a result, (1.4) holds for ℓ_2 .

Next we calculate the path-integrals of η and $g^2\eta$ along ℓ'_1 , and we want to reduce them to the path-integrals along ℓ_1 . Note that η has a pole at $(0, 0)$. To avoid a divergent integral, here we add an exact 1-form which has the principal part of η . It is straightforward to check

$$\eta - \frac{k+1}{2}d\left(\frac{z-1}{w}\right) = -\frac{k}{2}\frac{z-1}{w(z-a)}dz + \frac{1}{2}\frac{dz}{w}.$$

Thus we have

$$\begin{aligned}\oint_{\ell'_1} \eta &= \oint_{\ell'_1} \left(-\frac{k}{2}\frac{z-1}{w(z-a)}dz + \frac{1}{2}\frac{dz}{w}\right) \\ &= \oint_{\ell_1} \left(-\frac{k}{2}\frac{z-1}{w(z-a)}dz + \frac{1}{2}\frac{dz}{w}\right) = ie^{-\pi i/(k+1)} \sin \frac{\pi}{k+1} (kA_1 - A_2),\end{aligned}$$

where

$$A_1 = \int_0^1 \frac{(1-t)^{1/(k+1)}}{t^{2/(k+1)}(a-t)^{1/(k+1)}} dt, \quad A_2 = \int_0^1 \frac{(a-t)^{k/(k+1)}}{t^{2/(k+1)}(1-t)^{k/(k+1)}} dt.$$

Also, we have

$$\oint_{\ell'_1} g^2\eta = \oint_{\ell_1} g^2\eta = 2ie^{\pi i/(k+1)} \sin \frac{\pi}{k+1} a^{(k-2)/(k+1)} A_3,$$

where

$$A_3 = \int_0^1 \frac{(1-t)^{k/(k+1)}}{t^{(k-1)/(k+1)}(a-t)^{k/(k+1)}} dt.$$

Hence for the loop $\ell'_1 \in \pi_1(M)$, (1.4) holds if and only if

$$kA_1 + 2a^{(k-2)/(k+1)}A_3 - A_2 = 0. \quad (2.3)$$

Now we evaluate the values A_1 , A_2 , and A_3 . Since $1/a \leq 1/(a-t) \leq 1/(a-1)$, we see that

$$\begin{aligned}\frac{1}{a^{1/(k+1)}}B\left(\frac{k-1}{k+1}, \frac{k+2}{k+1}\right) &\leq A_1 \leq \frac{1}{(a-1)^{1/(k+1)}}B\left(\frac{k-1}{k+1}, \frac{k+2}{k+1}\right), \\ \frac{1}{a^{k/(k+1)}}B\left(\frac{2}{k+1}, \frac{2k+1}{k+1}\right) &\leq A_3 \leq \frac{1}{(a-1)^{k/(k+1)}}B\left(\frac{2}{k+1}, \frac{2k+1}{k+1}\right),\end{aligned}$$

where $B(x, y)$ is the beta function defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad (\operatorname{Re} x > 0, \operatorname{Re} y > 0).$$

Also, since $a-1 \leq a-t \leq a$, we have

$$(a-1)^{k/(k+1)}B\left(\frac{k-1}{k+1}, \frac{1}{k+1}\right) \leq A_2 \leq a^{k/(k+1)}B\left(\frac{k-1}{k+1}, \frac{1}{k+1}\right).$$

It follows that for the case $a \rightarrow \infty$, we have $A_1 \rightarrow 0$, $a^{(k-2)/(k+1)}A_3 \rightarrow 0$, and $A_2 \rightarrow \infty$. As a result, the left hand side of (2.3) is negative. On the other hand, for the case $a \rightarrow 1$, we have

$$\begin{aligned}
& kA_1 + 2a^{(k-2)/(k+1)}A_3 - A_2 \\
& \geq \frac{k}{a^{1/(k+1)}}B\left(\frac{k-1}{k+1}, \frac{k+2}{k+1}\right) + \frac{2}{a^{2/(k+1)}}B\left(\frac{2}{k+1}, \frac{2k+1}{k+1}\right) \\
& \quad - a^{k/(k+1)}B\left(\frac{k-1}{k+1}, \frac{1}{k+1}\right) \\
& = \frac{1}{a^{1/(k+1)}}B\left(\frac{k-1}{k+1}, \frac{1}{k+1}\right) + \frac{2}{a^{2/(k+1)}}B\left(\frac{2}{k+1}, \frac{2k+1}{k+1}\right) \\
& \quad - a^{k/(k+1)}B\left(\frac{k-1}{k+1}, \frac{1}{k+1}\right) \\
& \xrightarrow{a \rightarrow 1} 2B\left(\frac{2}{k+1}, \frac{2k+1}{k+1}\right) > 0,
\end{aligned}$$

and here we use the following formula for the beta function:

$$B(x, y+1) = \frac{y}{x+y}B(x, y).$$

So, the left hand side of (2.3) is positive.

Therefore, the intermediate value theorem yields that there exists $a \in (1, \infty)$ which satisfies (2.3). Moreover, since all of A_1 , $a^{(k-2)/(k+1)}A_3$, and $-A_2$ are monotone decreasing functions with respect to a , the left hand side of (2.3) is monotone decreasing function as well. This proves the uniqueness. (See Figure 2.5). \square

Since $\deg(g) = \gamma + 2$, the next corollary follows:

Corollary 2.6. *For all even number γ , there exists a complete conformal minimal surface of genus γ with two ends which has least total absolute curvature.*

Combining Corollaries 2.3 and 2.6 proves Main Theorem 1 in the introduction.

Next we discuss the above minimal surfaces from the point of view of the Björling problem. As we mentioned in the introduction, there is a construction method for minimal surfaces from a given curve. We show that every minimal surface given in this subsection gives a solution to the Björling problem in the higher genus case and the generating curve is a closed planar geodesic.

Let l be a fixed point set of $\kappa_3 \circ \kappa_1$. Using (1.1), we see that $\kappa_3 \circ \kappa_1$ is an isometry, and thus l is a geodesic. An explicit description of l is given by

$$\frac{a}{\bar{z}} = z, \quad \frac{1}{c^2 \bar{w}} = w,$$

that is,

$$|z| = \sqrt{a}, \quad |w| = \frac{1}{c}.$$

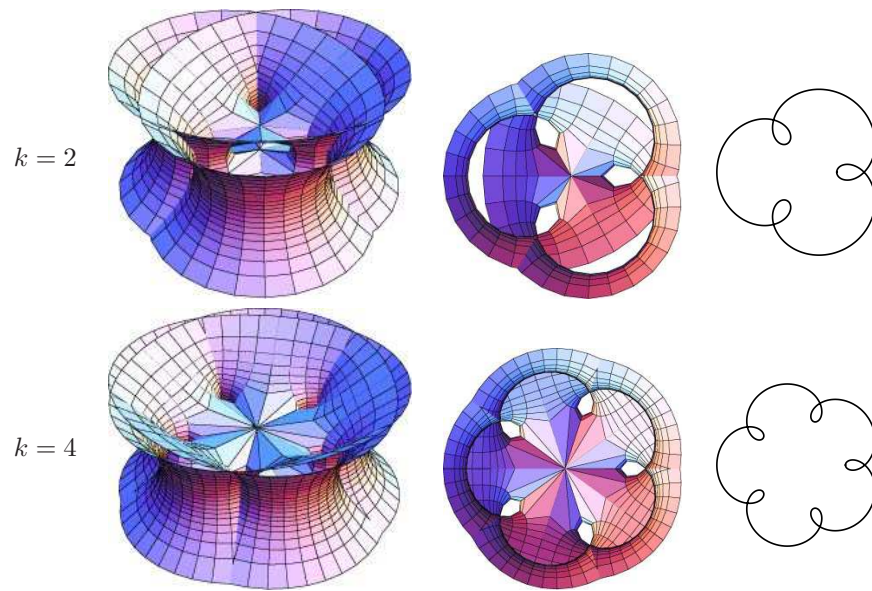


Figure 2.5: Minimal surfaces of genus k with two ends which satisfy $\deg(g) = k+2$. The middle columns show a half cut away of the surfaces by the xy -plane. The right columns show their intersection with the xy -plane.

Hence we conclude that l is a closed geodesic. Moreover, by Lemma 2.4, l lies in the xy -plane, and therefore the assertion follows.

3 Uniqueness

In this section, we will prove Main Theorem 2 through four subsections.

3.1 Symmetry

First, we refer to some basic results about symmetries of a minimal surface. (See p. 349 in [LM].)

Let $f : M \rightarrow \mathbb{R}^3$ be a conformal minimal immersion, with (g, η) its Weierstrass data. Suppose that $A : M \rightarrow M$ is a diffeomorphism. A is said to be a *symmetry* if there exists $O \in \mathcal{O}(3, \mathbb{R})$ and $v \in \mathbb{R}^3$ such that

$$(f \circ A)(p) = Of(p) + v.$$

Denote by $\text{Sym}(M)$ the group of symmetries of M , and by $\text{Iso}(M)$ the isometry group of M . Then, by definition, $\text{Sym}(M)$ is a subgroup of $\text{Iso}(M)$. Let $L(M)$ be the group of holomorphic and antiholomorphic diffeomorphisms α of M satisfying

$$G \circ \alpha(p) = O \circ G(p),$$

where $G : M \rightarrow S^2$ is the Gauss map and $O \in \mathcal{O}(3, \mathbb{R})$ is a linear isometry of \mathbb{R}^3 . We now assume that f is complete, and of finite total curvature. Lopez and Martin pointed out that if one of the following three differentials $(1 - g^2)\eta$, $i(1 + g^2)\eta$, $2g\eta$ is not exact, then

$$L(M) = \text{Iso}(M) = \text{Sym}(M).$$

Suppose that f has two ends. By Theorem 1.4, there exists a compact Riemann surface \overline{M}_γ of genus γ and two points $p_1, p_2 \in \overline{M}_\gamma$ such that M is conformally equivalent to $\overline{M}_\gamma - \{p_1, p_2\}$. A symmetry of $f(M)$ extends to \overline{M}_γ leaving the set $\{p_1, p_2\}$ invariant. By the Hurwitz' Theorem, the group $\text{Sym}(M)$ is finite, and so up to a suitable choice of the origin, $\text{Sym}(M)$ is a finite group Δ of orthogonal linear transformations of \mathbb{R}^3 .

We assume that $\text{Sym}(M)$ has $4(\gamma + 1)$ elements ($\gamma \geq 1$) and $L(M) = \text{Iso}(M) = \text{Sym}(M)$. If there is no symmetry in Δ such that either p_1 or p_2 is fixed, then Δ has at most 4 elements by a fundamental argument in linear algebra. Hence, we may assume without loss of generality that there exists a symmetry such that p_1 is fixed by the symmetry. Up to rotations, we may assume $g(p_1) = 0$, and then Δ leaves the x_3 -axis invariant.

We now focus on the following two cases: the case $\gamma = 1$ with $(d_1, d_2) = (1, 3)$, the even genus case with $(d_1, d_2) = (2, 2)$ (for the definition of d_i , see Equation (1.6)). For the former case, every symmetry in Δ leaves p_i invariant. So we see $g(p_2) = 0$ or ∞ . For the latter case, we have $|\Delta| \geq 12$, and then there exist at least two symmetries which leave p_i invariant. Hence $g(p_2) = 0$ or ∞ .

Let Δ_0 be the subgroup of holomorphic transformations in Δ , and denote by $\mathcal{R} \subset \Delta_0$ the cyclic subgroup of rotations around the x_3 -axis. Clearly, we obtain that

$$[\Delta : \Delta_0] \leq 2, \quad [\Delta_0 : \mathcal{R}] \leq 2. \quad (3.1)$$

So the subgroups $\Delta_0 \subset \Delta$ and $\mathcal{R} \subset \Delta_0$ are both normal.

Let R be the rotation around the x_3 -axis with the smallest positive angle in Δ_0 , that is, $\mathcal{R} = \langle R \rangle$. We first consider the quotient map $\pi_{\mathcal{R}} : \overline{M}_{\gamma} \rightarrow \overline{M}_{\gamma}/\mathcal{R}$. From (3.1), we see that

$$\deg(\pi_{\mathcal{R}}) = |\mathcal{R}| \geq \gamma + 1. \quad (3.2)$$

By the Riemann-Hurwitz formula, we have

$$\begin{aligned} |\mathcal{R}|(2 - 2\gamma(\overline{M}_{\gamma}/\mathcal{R})) &= 2 - 2\gamma + \sum_{p \in \overline{M}_{\gamma}} (\mu(p) - 1) \\ &= 2 - 2\gamma + 2(|\mathcal{R}| - 1) + \sum_{p \in M} (\mu(p) - 1), \end{aligned} \quad (3.3)$$

where $\gamma(\overline{M}_{\gamma}/\mathcal{R})$ is the genus of $\overline{M}_{\gamma}/\mathcal{R}$ and $\mu(p) - 1$ is the ramification index at p . Let q'_1, \dots, q'_t be ramified values of $\pi_{\mathcal{R}}$ except for the $\pi_{\mathcal{R}}(p_i)$'s, and $m_i - 1$ the ramification index at $p \in \pi_{\mathcal{R}}^{-1}(q'_i)$. Note that $2 \leq m_i \leq |\mathcal{R}|$ and the ramification index at p_i is $|\mathcal{R}| - 1$. Combining (3.2) and (3.3) yields

$$\begin{aligned} |\mathcal{R}|(2 - 2\gamma(\overline{M}_{\gamma}/\mathcal{R})) &= 2 - 2\gamma + 2(|\mathcal{R}| - 1) + \sum_{i=1}^t (m_i - 1) \frac{|\mathcal{R}|}{m_i} \\ &\geq 2 + \sum_{i=1}^t (m_i - 1) \frac{|\mathcal{R}|}{m_i} > 0. \end{aligned} \quad (3.4)$$

It follows that $\gamma(\overline{M}_{\gamma}/\mathcal{R}) = 0$. This, combined with (3.2) and (3.4), implies

$$2\gamma = \sum_{i=1}^t (m_i - 1) \frac{|\mathcal{R}|}{m_i} = |\mathcal{R}| \left\{ \sum_{i=1}^t \left(\frac{1}{2} - \frac{1}{m_i} \right) + \frac{t}{2} \right\} \geq (\gamma + 1) \frac{t}{2}. \quad (3.5)$$

So $t \leq \frac{4\gamma}{\gamma+1} < 4$, and thus $t = 1, 2, 3$. We remark that $|\mathcal{R}| = \gamma + 1, 2(\gamma + 1), 4(\gamma + 1)$.

The case $t = 1$

From the first equality of (3.5), we obtain $2\gamma = |\mathcal{R}|(1 - \frac{1}{m_1})$. For the case $|\mathcal{R}| = \gamma + 1$, $\frac{1}{m_1} = -\frac{\gamma-1}{\gamma+1} \leq 0$, which is absurd. Next, for the case $|\mathcal{R}| = 2(\gamma + 1)$, $m_1 = \gamma + 1$ holds. Finally, $|\mathcal{R}| = 4(\gamma + 1)$ gives $\frac{1}{2} > \frac{\gamma}{2(\gamma+1)} = 1 - \frac{1}{m_1} \geq 1 - \frac{1}{2} = \frac{1}{2}$ which leads to a contradiction.

The case $t = 2$

We obtain $2\gamma = |\mathcal{R}|(2 - \frac{1}{m_1} - \frac{1}{m_2})$ from the first equality of (3.5). Without loss of generality, we may assume $m_1 \leq m_2$. Then,

$$2 - \frac{2}{m_1} \leq \frac{2\gamma}{|\mathcal{R}|} = 2 - \frac{1}{m_1} - \frac{1}{m_2} \leq 2 - \frac{2}{m_2}.$$

For the case $|\mathcal{R}| = \gamma + 1$, $\gamma + 1 \leq m_2 \leq |\mathcal{R}| = \gamma + 1$ holds, and thus $m_2 = \gamma + 1$ and $m_1 = \gamma + 1$. Next we consider the case $|\mathcal{R}| \geq 2(\gamma + 1)$. In this case, we have $2 - \frac{2}{m_1} \leq \frac{2\gamma}{|\mathcal{R}|} \leq \frac{\gamma}{\gamma+1} < 1$, and so $m_1 < 2$, which is absurd.

The case $t = 3$

It follows from the first equality of (3.5) that $2\gamma = |\mathcal{R}|(3 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3})$. We may assume $m_1 \leq m_2 \leq m_3$. Then

$$3 - \frac{3}{m_1} \leq \frac{2\gamma}{|\mathcal{R}|} = 3 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3} \leq 3 - \frac{3}{m_3}.$$

For the case $|\mathcal{R}| = \gamma + 1$, $3 - \frac{3}{m_1} \leq \frac{2\gamma}{|\mathcal{R}|} = \frac{2\gamma}{\gamma+1} < 2$. So $m_1 < 3$, and hence $m_1 = 2$. As a result,

$$\frac{5}{2} - \frac{2}{m_2} \leq \frac{2\gamma}{\gamma+1} = \frac{5}{2} - \frac{1}{m_2} - \frac{1}{m_3} \leq \frac{5}{2} - \frac{2}{m_3}.$$

$\frac{5}{2} - \frac{2}{m_2} \leq \frac{2\gamma}{\gamma+1}$ gives $m_2 \leq \frac{4(\gamma+1)}{\gamma+5} < 4$. Thus $m_2 = 2, 3$. For the case $m_2 = 2$, $\frac{2\gamma}{\gamma+1} = 2 - \frac{1}{m_3}$ holds, and so $m_3 = \frac{\gamma+1}{2}$. For the case $m_2 = 3$, $\frac{2\gamma}{\gamma+1} = \frac{13}{6} - \frac{1}{m_3}$. It follows that $m_3 = \frac{6(\gamma+1)}{\gamma+13} < 6$, and hence $m_3 = 3, 4, 5$. So $\gamma = 11, 23, 59$. For the case $|\mathcal{R}| \geq 2(\gamma + 1)$, $3 - \frac{3}{m_1} \leq \frac{2\gamma}{|\mathcal{R}|} \leq \frac{\gamma}{\gamma+1} < 1$. So $m_1 < \frac{3}{2}$, and this contradicts $m_1 \geq 2$.

As a consequence, we obtain the following tables:

$ \mathcal{R} $	m_i
$ \mathcal{R} = \gamma + 1$	do not occur
$ \mathcal{R} = 2(\gamma + 1)$	$m_1 = \gamma + 1$
$ \mathcal{R} = 4(\gamma + 1)$	do not occur

Table 3.1: The case $t = 1$.

$ \mathcal{R} $	m_i
$ \mathcal{R} = \gamma + 1$	$m_1 = m_2 = \gamma + 1$
$ \mathcal{R} \geq 2(\gamma + 1)$	do not occur

Table 3.2: The case $t = 2$.

$ \mathcal{R} $	m_i			
	m_1	m_2	m_3	γ
$ \mathcal{R} = \gamma + 1$	2	2	$(\gamma + 1)/2$	odd (> 1)
	2	3	3	11
	2	3	4	23
	2	3	5	59
$ \mathcal{R} \geq 2(\gamma + 1)$	do not occur			

Table 3.3: The case $t = 3$.

Note that, for $t = 2$, $\pi_{\mathcal{R}}$ is a cyclic branched cover of S^2 , of order $\gamma + 1$, whose branch points are the fixed points of \mathcal{R} , that is, $p_1, p_2, \pi_{\mathcal{R}}^{-1}(q'_1), \pi_{\mathcal{R}}^{-1}(q'_2)$.

For the case $t = 1$, $\pi_{\mathcal{R}}^{-1}(q'_1) = \{q_1, q_2\}$ for some two points $q_1, q_2 \in \overline{M}_{\gamma}$. If R leaves every q_i invariant, then m_1 must be $2(\gamma + 1)$. Thus $R(q_1) = q_2$ and $R(q_2) = q_1$. So $R^2(q_i) = q_i$ and $f(q_1) = f(q_2) \in \{x_3\text{-axis}\}$. Now we consider the quotient map $\pi_{\langle R^2 \rangle} : \overline{M}_{\gamma} \rightarrow \overline{M}_{\gamma}/\langle R^2 \rangle$. From the Riemann-Hurwitz formula,

$$|\langle R^2 \rangle| (2 - 2\gamma(\overline{M}/\langle R^2 \rangle)) = 2 - 2\gamma + 4(|\langle R^2 \rangle| - 1) = 2\gamma + 2 > 0.$$

Hence we obtain $\gamma(\overline{M}/\langle R^2 \rangle) = 0$. It follows that $\pi_{\langle R^2 \rangle}$ is a cyclic branched cover of S^2 , of order $\gamma + 1$, whose branch points are p_1, p_2, q_1, q_2 . This case corresponds to the case $t = 2$, and thus we can determine the case $t = 1$ after we consider the case $t = 2$.

Next, we consider the quotient map $\pi_{\Delta_0} : \overline{M}_{\gamma} \rightarrow \overline{M}_{\gamma}/\Delta_0$ and repeat similar arguments as above. From the Riemann-Hurwitz formula, we obtain

$$2\gamma - 2 = |\Delta_0|(2\gamma(\overline{M}_{\gamma}/\Delta_0) - 2) + \sum_{p \in \overline{M}_{\gamma}} (\mu(p) - 1). \quad (3.6)$$

We now treat the two cases that there is a symmetry $\sigma \in \Delta_0$ satisfying $\sigma(p_1) = p_2$ or not. For our case, we may exclude the case $t = 3$, and consider the case $t = 2$. It follows from (3.1) that $|\Delta_0| = 2(\gamma + 1)$ and $\sigma \in \Delta_0 \setminus \mathcal{R}$.

The case p_1 can be transformed to p_2

If there exists such σ , then the ramification index at p_i must be $|\Delta_0|/2 - 1$. So (3.6) can be reduced to

$$2\gamma - 2 = |\Delta_0|(2\gamma(\overline{M}/\Delta_0) - 2) + 2(|\Delta_0|/2 - 1) + \sum_{p \in M} (\mu(p) - 1).$$

Hence,

$$\begin{aligned}
2\gamma &= |\Delta_0|(2\gamma(\overline{M}/\Delta_0) - 1) + \sum_{p \in M} (\mu(p) - 1) \\
&= 2(\gamma + 1)(2\gamma(\overline{M}/\Delta_0) - 1) + \sum_{p \in M} (\mu(p) - 1) \\
&\geq 2(\gamma + 1)(2\gamma(\overline{M}/\Delta_0) - 1).
\end{aligned} \tag{3.7}$$

So the case $\gamma(\overline{M}/\Delta_0) > 0$ leads to a contradiction, and thus $\gamma(\overline{M}/\Delta_0) = 0$ holds. As a consequence, (3.7) can be reduced to

$$4\gamma + 2 = \sum_{p \in M} (\mu(p) - 1). \tag{3.8}$$

Let r'_1, \dots, r'_s be ramified values of π_{Δ_0} except for the $\pi_{\Delta_0}(p_i)$'s, and $m_i - 1$ the ramification index at $p \in \pi_{\Delta_0}^{-1}(r'_i)$. Note that $2 \leq m_i \leq |\Delta_0|$. (3.8) can be rewritten as

$$4\gamma + 2 = \sum_{i=1}^s (m_i - 1) \frac{|\Delta_0|}{m_i} = 2(\gamma + 1) \sum_{i=1}^s \left(1 - \frac{1}{m_i}\right). \tag{3.9}$$

If $s = 1$, then, (3.9) yields $2\gamma < 0$, which is absurd. Hence $s \geq 2$, and (3.9) takes the form

$$2\gamma = 2(\gamma + 1) \left\{ \sum_{i=1}^s \left(1 - \frac{1}{m_i}\right) - 1 \right\} = 2(\gamma + 1) \left\{ \frac{s-2}{2} + \sum_{i=1}^s \left(\frac{1}{2} - \frac{1}{m_i}\right) \right\}.$$

So

$$1 > \frac{2\gamma}{2(\gamma + 1)} = \frac{s-2}{2} + \sum_{i=1}^s \left(\frac{1}{2} - \frac{1}{m_i}\right) \geq \frac{s-2}{2}.$$

As a result, $2 \leq s < 4$ follows, and thus $s = 2, 3$.

The case $s = 2$
(3.9) implies

$$2\gamma = 2(\gamma + 1) \left(1 - \frac{1}{m_1} - \frac{1}{m_2}\right),$$

that is,

$$\frac{1}{m_1} + \frac{1}{m_2} = \frac{1}{\gamma + 1}$$

holds. The inequalities $2 \leq m_i \leq |\Delta_0| = 2(\gamma + 1)$ yield $m_1 = m_2 = 2(\gamma + 1)$.

The case $s = 3$
From (3.9),

$$2\gamma = 2(\gamma + 1) \left(2 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3}\right). \tag{3.10}$$

Without loss of generality, we may assume $m_1 \leq m_2 \leq m_3$. In this case,

$$2 - \frac{3}{m_1} \leq 2 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3} \leq 2 - \frac{3}{m_3}.$$

By (3.10), we obtain

$$2(\gamma + 1) \left(2 - \frac{3}{m_1} \right) \leq 2(\gamma + 1) \left(2 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3} \right) = 2\gamma.$$

Thus we have

$$m_1 \leq \frac{3(\gamma + 1)}{\gamma + 2} < 3.$$

Hence $m_1 = 2$. Moreover, let us consider the case $m_1 = 2$ and $m_2 \leq m_3$. Then

$$2(\gamma + 1) \left(\frac{3}{2} - \frac{2}{m_2} \right) \leq 2(\gamma + 1) \left(2 - \frac{1}{2} - \frac{1}{m_2} - \frac{1}{m_3} \right) = 2\gamma,$$

and so

$$m_2 \leq \frac{4(\gamma + 1)}{\gamma + 3} < 4.$$

It follows that $m_2 = 2, 3$. For the case $m_2 = 2$, $2\gamma = 2(\gamma + 1) \left(1 - \frac{1}{m_3} \right)$, and thus $m_3 = \gamma + 1$. For the case $m_2 = 3$, $2\gamma = 2(\gamma + 1) \left(\frac{7}{6} - \frac{1}{m_3} \right)$ and so $m_3 = \frac{6(\gamma + 1)}{\gamma + 7} < 6$. As a consequence, $(m_3, \gamma) = (3, 5), (4, 11), (5, 29)$.

The case p_1 cannot be transformed to p_2

If there does not exist σ satisfying $\sigma(p_1) = p_2$, then the ramification index at p_i must be $|\Delta_0| - 1$. It follows that (3.6) can be reduced to

$$2\gamma - 2 = |\Delta_0|(2\gamma(\overline{M}/\Delta_0) - 2) + 2(|\Delta_0| - 1) + \sum_{p \in M} (\mu(p) - 1),$$

and thus

$$2\gamma = 2|\Delta_0|\gamma(\overline{M}/\Delta_0) + \sum_{p \in M} (\mu(p) - 1) = 4(\gamma + 1)\gamma(\overline{M}/\Delta_0) + \sum_{p \in M} (\mu(p) - 1). \quad (3.11)$$

(3.11) yields $\gamma(\overline{M}/\Delta_0) = 0$, and so

$$2\gamma = \sum_{p \in M} (\mu(p) - 1). \quad (3.12)$$

Suppose that r'_1, \dots, r'_s are ramified values of π_{Δ_0} except for the $\pi_{\Delta_0}(p_i)$'s, and $m_i - 1$ the ramification index at $p \in \pi_{\Delta_0}^{-1}(r'_i)$ as above. (3.12) can be rewritten as

$$2\gamma = \sum_{i=1}^s (m_i - 1) \frac{|\Delta_0|}{m_i} = 2(\gamma + 1) \sum_{i=1}^s \left(1 - \frac{1}{m_i} \right).$$

As a result,

$$1 > \frac{2\gamma}{2\gamma+2} = \sum_{i=1}^s \left(1 - \frac{1}{m_i}\right) = \sum_{i=1}^s \left\{ \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{m_i}\right) \right\} \geq \frac{s}{2},$$

and hence $1 \leq s < 2$, that is, $s = 1$. Thus, we have $m_1 = \gamma + 1$.

Therefore, we obtain the following tables:

s		m_i		
$s = 2$		$m_1 = m_2 = 2(\gamma + 1)$		
$s = 3$	m_1	m_2	m_3	γ
	2	2	$\gamma + 1$	arbitrary
	2	3	3	5
	2	3	4	11
	2	3	5	29

Table 3.4: The case p_1 can be transformed to p_2 .

s	m_i
$s = 1$	$m_1 = \gamma + 1$

Table 3.5: The case p_1 cannot be transformed to p_2 .

3.2 Weierstrass data for the case $\gamma = 1$ with $(d_1, d_2) = (1, 3)$

By Tables 3.1–3.3, we first consider the case $t = 2$. Then $|\mathcal{R}| = 2$ and we find $|\Delta_0| = 4$ by (3.1). Set q_1, q_2 as two branch points of $\pi_{\mathcal{R}}$ distinct from the p_i 's and $p'_i := \pi_{\mathcal{R}}(p_i)$, $q'_i := \pi_{\mathcal{R}}(q_i)$. Since $\pi_{\mathcal{R}}$ is a cyclic branched double cover of S^2 , \overline{M}_1 can be given by

$$v^2 = (u - p'_1)^{m_1 h_1} (u - p'_2)^{m_2 h_2} (u - q'_1)^{m_3 h_3} (u - q'_2)^{m_4},$$

where $h_i \in \{1, -1\}$ ($i = 1, 2, 3$), $(2, m_i) = 1$, and $R(u, v) = (u, -v)$.

Since $(d_1, d_2) = (1, 3)$, there does not exist $\sigma \in \Delta_0$ satisfying $\sigma(p_1) = p_2$. By Table 3.5 on Δ_0 , there exists a transformation $\tau \in \Delta_0 \setminus \mathcal{R}$ satisfying $\tau(q_1) = q_2$, and thus $m_3 = m_4$. τ induces a degree 2 transformation $\tau' : \overline{M}_1/\mathcal{R} \rightarrow \overline{M}_1/\Delta_0$, that is, a transformation on S^2 such that $\tau'(q'_1) = \tau'(q'_2)$. Choosing suitable variables (z, w) , we can represent $\tau(z, w) = (-z, *)$, $\tau'(z) = z^2$, $p'_1 = 0$, $p'_2 = \infty$, $q'_1 = 1$, $q'_2 = -1$, and moreover, \overline{M}_1 can be rewritten as $w^2 = z(z^2 - 1)$ (see Figure 3.1).

Now we consider the Gauss map. p_1, p_2, q_1, q_2 are fixed points of \mathcal{R} , that is, fixed points by rotations around the x_3 -axis. So we have $g(\{p_1, p_2, q_1, q_2\}) \subset$

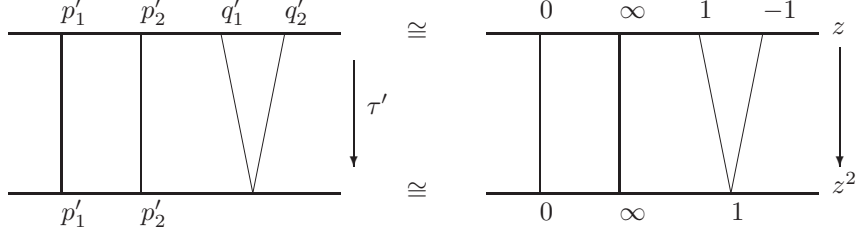


Figure 3.1: The Riemann surface \overline{M}_1 .

$\{0, \infty\}$. Note that q_1 can be transformed to q_2 by the biholomorphism τ . On the other hand, p_1 cannot be transformed to p_2 . It follows that we essentially only need to consider the two cases as in Figure 3.2:

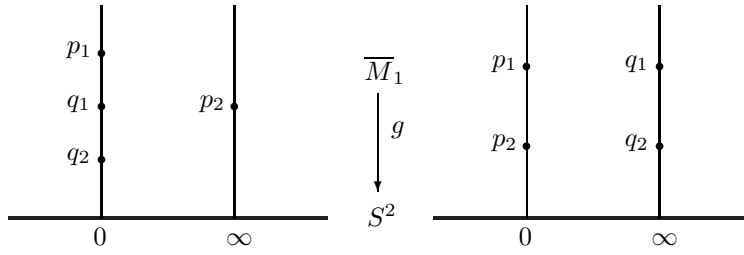


Figure 3.2: The possibilities of the Gauss map.

In the right hand side case in Figure 3.2, the ramification index at q_i is $\frac{\gamma+2}{2} - 1 \notin \mathbb{Z}$ since the ramification index at q_1 of g must coincide with the ramification index at q_2 of g . Hence we only consider the left hand side case in Figure 3.2. The ramification index at p_2 may be $1 - 1, 2 - 1, 3 - 1$. If the ramification index is $2 - 1$, $g^{-1}(\infty)$ consists of p_2 and a simple pole $q \in \overline{M}_1$. Then $R(q)$ must be a pole of g , but $R(q) \notin \{p_2, q\}$. This contradicts $R(q) \in g^{-1}(\infty)$. So the divisor of g is given by

$$(g) = \begin{cases} p_1 + q_1 + q_2 - 3p_2 \\ p_1 + q_1 + q_2 - p_2 - Q - R(Q) \end{cases}$$

for a point Q . Since $\tau(p_2) = p_2$, τ leaves $\{\text{the poles of } g\}$ invariant. For the latter case, if we take $Q^* := \tau(Q)$, then Q^* must be a pole of g which is distinct from the $R^i(Q)$'s. It leads to a contradiction, and so we only consider the former case. In this case, the divisor of the meromorphic function $z(z^2 - 1)$ coincides with that of g^2 . Thus $g^2 = c' z(z^2 - 1)$ holds for some constant c' . Hence, \overline{M}_1 and g can be rewritten as

$$w^2 = z(z^2 - 1), \quad g = c w$$

for some constant c , and $R(z, w) = (z, -w)$, $\tau(z, w) = (-z, iw)$. Then, the divisor of η is obtained by

$$(\eta) = \begin{cases} -2p_1 + 2p_2 \\ -4p_1 + 4p_2. \end{cases}$$

Thus, by a similar argument,

$$\eta^2 = \begin{cases} c'' \frac{(dz)^2}{z^3(z^2-1)} \\ c'' \frac{(dz)^2}{z^5(z^2-1)} \end{cases} = \begin{cases} c'' \left(\frac{dz}{zw} \right)^2 \\ c'' \left(\frac{dz}{z^2 w} \right)^2 \end{cases}$$

hold for some constant c'' . As a consequence, we obtain $\eta = c''' \frac{dz}{zw}$, $c''' \frac{dz}{z^2 w}$ for some constant c''' . The latter case is given in §2.1, and we shall prove that the former case does not occur in § 3.4. Note that the case $t = 1$ does not occur in this case.

3.3 Weierstrass data for the even genus case with $(d_1, d_2) = (2, 2)$

We treat the case $t = 2$ ($|\mathcal{R}| = \gamma + 1$, $|\Delta_0| = 2(\gamma + 1)$). By Table 3.2, $\pi_{\mathcal{R}} : \overline{M}_{\gamma} \rightarrow \overline{M}_{\gamma}/\mathcal{R}$ is a cyclic branched cover of S^2 . Thus \overline{M}_{γ} can be represented by

$$v^{\gamma+1} = (u - p'_1)^{m_1 h_1} (u - p'_2)^{m_2 h_2} (u - q'_1)^{m_3 h_3} (u - q'_2)^{m_4},$$

where $(m_i, \gamma + 1) = 1$, $h_i = \pm 1$, and $R(u, v) = \left(u, e^{\frac{2\pi}{\gamma+1}i} v\right)$.

The case p_1 cannot be transformed to p_2

We assume that there does not exist $\sigma \in \Delta_0$ such that $\sigma(p_1) = p_2$. From the table on Δ_0 , there exists a transformation $\tau \in \Delta_0 \setminus \mathcal{R}$ satisfying $\tau(q_1) = q_2$, and thus $m_3 = m_4$. τ induces a degree 2 transformation $\tau' : \overline{M}_1/\mathcal{R} \rightarrow \overline{M}_1/\Delta_0$, that is, a transformation on S^2 such that $\tau'(q'_1) = \tau'(q'_2)$. Choosing suitable variables (z, w) , we can represent $\tau(z, w) = (-z, *)$, $\tau'(z) = z^2$, $p'_1 = 0$, $p'_2 = \infty$, $q'_1 = 1$, $q'_2 = -1$, and moreover, \overline{M}_{γ} can be rewritten as $w^{\gamma+1} = z^{m_1 h_1} (z^2 - 1)^{m_3}$ and $R(z, w) = \left(z, e^{\frac{2\pi}{\gamma+1}i} w\right)$ (see Figure 3.3).

Now we consider the Gauss map. p_1, p_2, q_1, q_2 are fixed points of \mathcal{R} , that is, fixed points by rotations around the x_3 -axis. So we find $g(\{p_1, p_2, q_1, q_2\}) \subset \{0, \infty\}$. Note that q_1 can be transformed to q_2 by the biholomorphism τ . On the other hand, p_1 cannot be transformed to p_2 . It follows that we essentially only need to consider the two cases as in Figure 3.4:

The left hand side case in Figure 3.4

The divisor of g is given by

$$(g) = \begin{cases} (\gamma + 2 - 2N)p_1 + Nq_1 + Nq_2 - (\gamma + 2)p_2 \\ (\gamma + 2 - 2N)p_1 + Nq_1 + Nq_2 - p_2 - Q - R(Q) - \cdots - R^{\gamma}(Q) \end{cases}$$

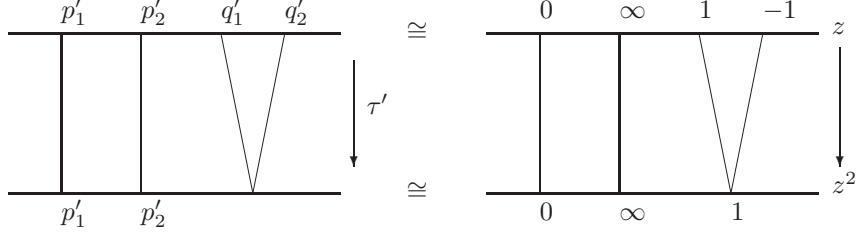


Figure 3.3: The Riemann surface \overline{M}_γ .

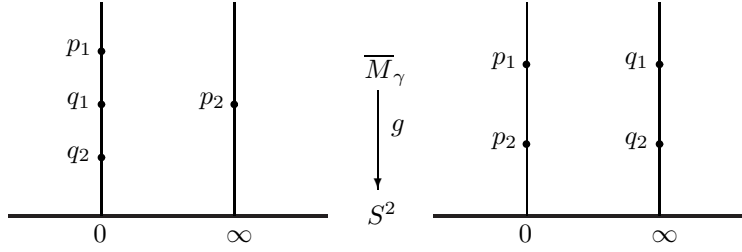


Figure 3.4: The possibilities of the Gauss map.

for a point Q . Note that $N > 0$ and $\gamma + 2 - 2N > 0$. Since $\tau(p_2) = p_2$, τ leaves $\{\text{the poles of } g\}$ invariant. For the latter case, if we take $Q^* := \tau(Q)$, then Q^* must be a pole of g which is distinct from the $R^i(Q)$'s. This leads to a contradiction, and so we only consider the former case. Then the divisor of η is given by

$$(\eta) = -3p_1 + (2\gamma + 1)p_2.$$

Hence the divisor of $g\eta$ is obtained by

$$(g\eta) = (\gamma - 2N - 1)p_1 + Nq_1 + Nq_2 + (\gamma - 1)p_2.$$

If $\gamma - 2N - 1 \geq 0$, then $g\eta$ is holomorphic. Thus f is bounded and this leads to a contradiction. As a result, $\gamma - 2N - 1 < 0$ follows. The inequality $\gamma + 2 - 2N > 0$ yields $N < \frac{\gamma}{2} + 1$. Also, since γ is even, $N \leq \frac{\gamma}{2}$ holds. So we have $\gamma = 2N$.

It follows that the divisor of $g^{\gamma+1}$ coincides with that of $z^2(z^2 - 1)^{\frac{\gamma}{2}}$. Therefore, \overline{M}_γ and g can be rewritten as

$$w^{\gamma+1} = z^2(z^2 - 1)^{\frac{\gamma}{2}}, \quad g = cw$$

for some constant c , and $R(z, w) = (z, e^{\frac{2\pi i}{\gamma+1}}w)$, $\tau(z, w) = (-z, w)$. Furthermore, the divisor of $\eta^{\gamma+1}$ coincides with that of $\frac{(dz)^{\gamma+1}}{z^{\gamma-1}g^{2(\gamma+1)}}$. Hence

$$\eta^{\gamma+1} = c'' \frac{(dz)^{\gamma+1}}{z^{\gamma-1}g^{2(\gamma+1)}} \left(= c'' z^2 \frac{(dz)^{\gamma+1}}{z^{\gamma+1}g^{2(\gamma+1)}} \right)$$

for some constant c'' . By setting $z = u^{\gamma+1}$ and $w = u^2v$, \overline{M}_γ can be rewritten as

$$v^{\gamma+1} = (u^{2(\gamma+1)} - 1)^{\frac{\gamma}{2}},$$

and moreover

$$g = cu^2v, \quad \eta = c''' \frac{u}{g^2} du$$

for some constant c''' . However, in this case, its genus is greater than γ , and such a case is excluded.

The right hand side case in Figure 3.4

The divisor of g is obtained by

$$(g) = (\gamma + 2 - N)p_1 + Np_2 - \frac{\gamma + 2}{2}q_1 - \frac{\gamma + 2}{2}q_2,$$

where $N > 0$ and $\gamma + 2 - N > 0$. Also, the divisor of η is given by

$$(\eta) = -3p_1 - 3p_2 + (\gamma + 2)q_1 + (\gamma + 2)q_2.$$

Thus the divisor of $g\eta$ is obtained by

$$(g\eta) = (\gamma - N - 1)p_1 + (N - 3)p_2 + \frac{\gamma + 2}{2}q_1 + \frac{\gamma + 2}{2}q_2.$$

If $\gamma - N - 1 \geq 0$ and $N - 3 \geq 0$, then $g\eta$ is holomorphic. Hence $\gamma - N - 1 < 0$ or $N - 3 < 0$. From the inequality $\gamma + 2 - N > 0$, the case $\gamma - N - 1 < 0$ corresponds to $\gamma = N$, $N - 1$. The case $N - 3 < 0$ implies $N = 1, 2$. Essentially, we consider the cases $N = 1, 2$. The divisor of $g^{\gamma+1}$ coincides with $\frac{z^{\gamma+2-N}}{(z^2 - 1)^{\frac{\gamma+2}{2}}}$.

As a consequence, \overline{M}_γ and g can be rewritten as

$$w^{\gamma+1} = \frac{(z^2 - 1)^{\frac{\gamma+2}{2}}}{z^{\gamma+2-N}}, \quad g = \frac{c}{w},$$

for some constant c . If $N = 1$, then $\gamma + 2 - N$ and $\gamma + 1$ are not coprime. So $N = 2$, and $R(z, w) = (z, e^{\frac{2\pi i}{\gamma+1}} w)$, $\tau(z, w) = (-z, w)$. Furthermore, the divisor $\eta^{\gamma+1}$ coincides with that of $\frac{(z^2 - 1)^2}{z^{\gamma+3}}(dz)^{\gamma+1}$. As a consequence,

$$\eta^{\gamma+1} = c' \frac{(z^2 - 1)^2}{z^{\gamma+3}}(dz)^{\gamma+1} \left(= \frac{c'}{z^6} \left(\frac{z^3 w^4 dz}{(z^2 - 1)^2} \right)^{\gamma+1} \right)$$

for some constant c' . By similar arguments as above, we may exclude this case except the case $\gamma = 2$. For the case $\gamma = 2$, \overline{M}_γ and g can be rewritten as

$$w^3 = \frac{(z^2 - 1)^2}{z^2}, \quad g = \frac{c}{w},$$

and moreover, we find $\eta = c'' \frac{w}{z} dz$ for some constants c, c'' . However, this surface has a transformation $\sigma \in \Delta_0$ defined by $\sigma(z, w) = (\frac{1}{z}, w)$, and we have $\sigma(p_1) = p_2$. This contradicts our assumption.

The case p_1 can be transformed to p_2

Suppose that there exists $\sigma \in \Delta_0$ such that $\sigma(p_1) = p_2$. By Table 3.4, we consider two cases, that is, the case $s = 2$ and the case $s = 3$. Note that $\sigma(p_2) = p_1$ and $\sigma \in \Delta_0 \setminus \mathcal{R}$.

The case $s = 2$

By Table 3.4, every q_i must be branch points of π_{Δ_0} with the ramified index $2(\gamma + 1) - 1$. Hence, $\sigma(q_i) = q_i$ for $i = 1, 2$, and moreover, σ induces a degree 2 transformation $\sigma' : \overline{M}_\gamma / \mathcal{R} \rightarrow \overline{M}_\gamma / \Delta_0$, that is a transformation on S^2 and the q_i 's are two fixed points of σ' (see Figure 3.5).

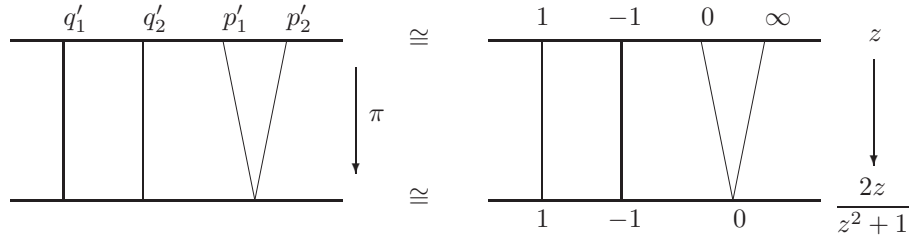


Figure 3.5: The Riemann surface \overline{M}_1 (for the case $s = 2$).

By suitable variables (z, w) , we have $\sigma(z, w) = \left(\frac{1}{z}, w\right)$, $\sigma'(z) = \frac{2z}{z^2 + 1}$, $p'_1 = 0, p'_2 = \infty, q'_1 = 1, q'_2 = -1$. Also, \overline{M}_γ is given by $w^2 = z^{m_1 h_1} (z - 1)^{m_2 h_2} (z + 1)^{m_3 h_3}$.

We consider the Gauss map. Essentially, we treat the two cases as in Figure 3.6:

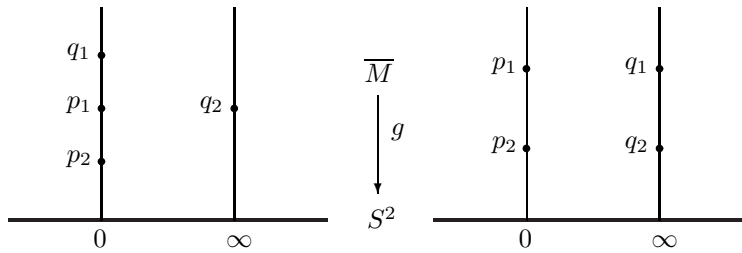


Figure 3.6: The possibilities of the Gauss map.

For the case in the left hand side of Figure 3.6, the divisor of g is given by

$$(g) = \begin{cases} (\gamma + 2 - 2N)q_1 + Np_1 + Np_2 - (\gamma + 2)q_2 \\ (\gamma + 2 - 2N)q_1 + Np_1 + Np_2 - q_2 - Q - R(Q) - \dots - R^\gamma(Q), \end{cases}$$

where $N > 0$ and $\gamma + 2 - 2N > 0$. Since $\sigma(q_2) = q_2$, σ leaves $\{\text{the poles of } g\}$ invariant. For the latter case, if we take $Q^* := \sigma(Q)$, then Q^* must be a pole of g which is distinct from the Q_i 's. This leads to a contradiction, and so we only consider the former case. In this case, we see

$$(\eta) = -3p_1 - 3p_2 + (2\gamma + 4)q_2.$$

Then

$$(g\eta) = (\gamma + 2 - 2N)q_1 + (N - 3)p_1 + (N - 3)p_2 + (\gamma + 2)q_2.$$

It follows that $N - 3 < 0$. Also $N > 0$ yields $N = 1, 2$.

Thus the divisor of $g^{\gamma+1}$ coincides with that of $\frac{z^N(z-1)^{\gamma+2-2N}}{(z+1)^{\gamma+2}}$. Therefore, \overline{M}_γ and g can be rewritten as

$$w^{\gamma+1} = \frac{z^N(z-1)^{\gamma+2-2N}}{(z+1)^{\gamma+2}} \quad (N = 1, 2), \quad g = cw$$

for some constant c , and $R(z, w) = (z, e^{\frac{2\pi}{\gamma+1}i}w)$, $\sigma(z, w) = (\frac{1}{z}, w)$. Furthermore, by similar arguments, η can be obtained by

$$\eta^{\gamma+1} = c' \frac{(z+1)^{\gamma+4}}{z^{\gamma+3}(z-1)^\gamma} (dz)^{\gamma+1} = \begin{cases} c' \frac{z+1}{z-1} \left(\frac{(z-1)dz}{z(z+1)w^2} \right)^{\gamma+1} & (N = 1) \\ c' \left(\frac{z+1}{z-1} \right)^2 \left(\frac{dz}{zw} \right)^{\gamma+1} & (N = 2) \end{cases}$$

for some constant c' . So we may exclude this case, like the previous case.

Next we consider the case in the right hand side of Figure 3.6. Then the divisor of g is obtained by

$$(g) = \frac{\gamma+2}{2}p_1 + \frac{\gamma+2}{2}p_2 - Nq_1 - (\gamma+2-N)q_2,$$

where $N > 0$ and $\gamma+2-N > 0$. In this case, the divisor of η is given by

$$(\eta) = -3p_1 - 3p_2 + 2Nq_1 + (2\gamma+4-2N)q_2.$$

Thus

$$(g\eta) = \frac{\gamma-4}{2}p_1 + \frac{\gamma-4}{2}p_2 + Nq_1 + (\gamma+2-N)q_2.$$

Hence $\gamma-4 < 0$ and so $\gamma = 2$. Moreover, from the inequality $\gamma+2-N > 0$, we obtain $N = 1, 2, 3$. If $N = 1$, then $\gamma+2-N$ and $\gamma+1$ are not coprime. Also, if $N =$

3, then N and $\gamma+1$ are not coprime. So we have $N = 2$. As a result, the divisor of $g^{\gamma+1}(=g^3)$ coincides with that of $\frac{z^{\frac{\gamma+2}{2}}}{(z-1)^N(z+1)^{\gamma+2-N}} \left(= \frac{z^2}{(z-1)^2(z+1)^2} \right)$. Therefore, \overline{M}_2 and g can be rewritten as

$$w^3 = \frac{(z-1)^2(z+1)^2}{z^2}, \quad g = \frac{c}{w}$$

for some constant c , and $R(z, w) = (z, e^{\frac{2\pi}{3}i}w)$, $\sigma(z, w) = (\frac{1}{z}, w)$. Also, we have $\eta = c' \frac{w}{z} dz$ for some constant c' . We shall prove that this case does not occur in §3.4.

The case $s = 3$

By Table 3.4, the p_i 's and q_i 's must be branch points of π_{Δ_0} with the ramified index $(\gamma+1) - 1$. As a result, there exist two sets $\{r_1^{(1)}, \dots, r_{\gamma+1}^{(1)}\}$, $\{r_1^{(2)}, \dots, r_{\gamma+1}^{(2)}\}$ of branch points with the ramified index 2 of π_{Δ_0} satisfying $\pi_{\Delta_0}(r_i^{(1)}) = \pi_{\Delta_0}(r_j^{(1)})$ and $\pi_{\Delta_0}(r_i^{(2)}) = \pi_{\Delta_0}(r_j^{(2)})$ for $1 \leq i, j \leq \gamma+1$. Note that $r_i^{(1)}$ and $r_i^{(2)}$ are distinct from the p_i 's and q_i 's. Hence, $\sigma(q_1) = q_2$ and $\sigma(q_2) = q_1$, and moreover, σ induces a degree 2 transformation $\sigma' : \overline{M}_\gamma/\mathcal{R} \rightarrow \overline{M}_\gamma/\Delta_0$, that is, a transformation on S^2 and $r'_i := \pi_{\mathcal{R}}(r_j^{(i)})$'s are two branch points of σ' (see Figure 3.7).

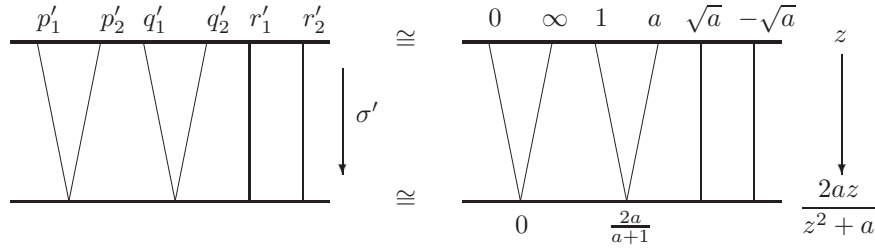


Figure 3.7: The Riemann surface \overline{M}_1 (the case $s = 3$).

Choosing suitable variables (z, w) , for $a \in \mathbb{C} \setminus \{0\}$, we have $\sigma(z, w) = \left(\frac{a}{z}, *\right) \sigma'(z) = \frac{2az}{z^2 + a}$, $p'_1 = 0$, $p'_2 = \infty$, $q'_1 = 1$, $q'_2 = a$, $r'_1 = \sqrt{a}$, $r'_2 = -\sqrt{a}$. Also, \overline{M}_γ is given by $w^{\gamma+1} = z^{m_1 h_1} (z-1)^{m_2 h_2} (z-a)^{m_2 h_3}$ (see Figure 3.7).

We now consider the Gauss map. Essentially, we consider the two cases as in Figure 3.8:

Then the divisor of g is obtained by

$$(g) = \begin{cases} (\gamma+2-N)p_1 + Nq_1 - (\gamma+2-N)p_2 - Nq_2 & \text{(the LHS case)} \\ \frac{\gamma+2}{2}p_1 + \frac{\gamma+2}{2}p_2 - \frac{\gamma+2}{2}q_1 - \frac{\gamma+2}{2}q_2 & \text{(the RHS case),} \end{cases}$$

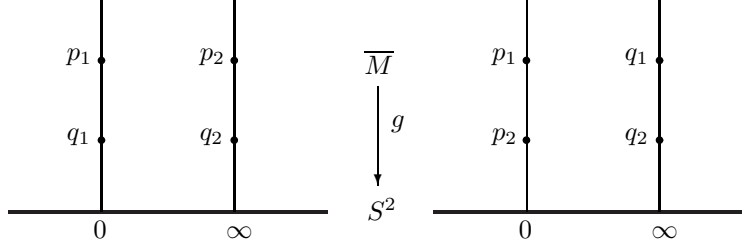


Figure 3.8: The possibilities of the Gauss map.

where $N > 0$ and $\gamma + 2 - N > 0$. The divisor of η is given by

$$(\eta) = \begin{cases} -3p_1 + (2\gamma - 2N + 1)p_2 + 2Nq_2 \\ -3p_1 - 3p_2 + (\gamma + 2)q_1 + (\gamma + 2)q_2. \end{cases}$$

So the divisor of $g\eta$ is obtained by

$$(g\eta) = \begin{cases} (\gamma - N - 1)p_1 + (\gamma - N - 1)p_2 + Nq_1 + Nq_2 \\ \frac{\gamma - 4}{2}p_1 + \frac{\gamma - 4}{2}p_2 + \frac{\gamma + 2}{2}q_1 + \frac{\gamma + 2}{2}q_2. \end{cases}$$

For the former case, if $\gamma - N - 1 \geq 0$, then $g\eta$ is holomorphic. Thus $\gamma - N - 1 < 0$ holds. Also, the inequality $\gamma + 2 - N > 0$ yields $N = \gamma, \gamma + 1$. For the latter case, $\gamma - 4 \geq 0$ cannot hold, and hence $\gamma = 2$. As a consequence,

$$g^{\gamma+1} = \begin{cases} c' z^{\gamma+2-N} \left(\frac{z-1}{z-a} \right)^N & \text{(the LHS case)} \\ c' \left(\frac{z}{(z-1)(z-a)} \right)^{\frac{\gamma+2}{2}} & \text{(the RHS case)} \end{cases}$$

for some constant c' . If $N = \gamma + 1$, then $\gamma + 1$ and N are not coprime. Thus $N = \gamma$ follows. Therefore, \overline{M}_γ and g can be rewritten as

$$\begin{cases} w^{\gamma+1} = z^2 \left(\frac{z-1}{z-a} \right)^\gamma, & g = cw & \text{(the LHS case)} \\ w^3 = \left(\frac{(z-1)(z-a)}{z} \right)^2, & g = \frac{c}{w} & \text{(the RHS case)} \end{cases}$$

and

$$R(z, w) = (z, e^{\frac{2\pi}{\gamma+1}i}w), \quad \sigma(z, w) = \begin{cases} \left(\frac{a}{z}, \frac{a^{\frac{\gamma+2}{\gamma+1}}}{w} \right) & \text{(the LHS case)} \\ \left(\frac{a}{z}, w \right) & \text{(the RHS case)}. \end{cases}$$

Also, for some constant c' , we have

$$\eta = \begin{cases} c' \frac{dz}{zw} & (\text{the LHS case}) \\ c' \frac{w}{z} dz & (\text{the RHS case}). \end{cases}$$

$\Delta \setminus \Delta_0 \neq \emptyset$ implies that there exists a degree 2 antiholomorphic transformation. Hence $a \in \mathbb{R}$ and the antiholomorphic transformation can be represented by $(z, w) \mapsto (\bar{z}, \bar{w})$. The former case corresponds to our result in §2.2, and the latter case is considered in §3.4. Note that the case $t = 1$ does not occur.

3.4 Well-definedness

In this subsection, we consider the well-definedness for the following two cases:

\overline{M}_γ	g	η	symmetries
$w^2 = z(z^2 - 1)$	cw	$c' \frac{dz}{zw}$	$R(z, w) = (-z, iw)$
$w^3 = \frac{(z-1)^2(z-a)^2}{z^2}$	$\frac{c}{w}$	$c' \frac{w}{z} dz$	$R(z, w) = (z, e^{\frac{2\pi}{3}i}w), \sigma(z, w) = (\frac{a}{z}, w)$

Note that $c, c' \in \mathbb{C} \setminus \{0\}$, $a \in \mathbb{R} \setminus \{0, 1\}$. M is given by

$$M = \begin{cases} \overline{M}_1 \setminus \{(0, 0), (\infty, \infty)\} & (\text{the former case}) \\ \overline{M}_2 \setminus \{(0, \infty), (\infty, \infty)\} & (\text{the latter case}). \end{cases}$$

The case $a = -1$ corresponds to the surface which we treat for the case $s = 2$. Note that the Weierstrass data $(e^{i\theta}g, e^{-i\theta}\eta)$ produces the same minimal surface as (g, η) rotated by an angle θ around the x_3 -axis. So after a suitable rotation of the surface, we may assume $c \in \mathbb{R}_+$. Also, multiplying a positive real number into η is just a homothety, so we may assume that $|c'| = 1$.

Our claim is that all cases do not occur.

The former case

First we consider $\Phi = {}^t(\Phi_1, \Phi_2, \Phi_3)$ in Theorem 1.1:

$$\Phi_1 = \left(\frac{1}{w} - c^2w\right) c' \frac{dz}{z}, \quad \Phi_2 = i \left(\frac{1}{w} + c^2w\right) c' \frac{dz}{z}, \quad \Phi_3 = 2cc' \frac{dz}{z}.$$

For the residue of Φ_3 at $z = 0$ to be real, we see that $c' = \pm 1$. We may choose $c' = 1$. We shall use the notation in the proof of Theorem 2.2 for $\gamma = 1$.

Straightforward calculation yields

$$\frac{dz}{zw} - d\left(\frac{2w}{z}\right) = -\frac{z}{w} dz.$$

Thus we have

$$\begin{aligned}\oint_{\ell'} \eta &= - \oint_{\ell'} \frac{z}{w} dz = - \oint_{\ell} \frac{z}{w} dz = -2i \int_0^1 \sqrt{\frac{t}{1-t^2}} dt, \\ \oint_{\ell'} g^2 \eta &= c^2 \oint_{\ell} \frac{w}{z} dz = -2ic^2 \int_0^1 \sqrt{\frac{1-t^2}{t}} dt.\end{aligned}$$

(1.4) implies

$$- \int_0^1 \sqrt{\frac{t}{1-t^2}} dt = c^2 \int_0^1 \sqrt{\frac{1-t^2}{t}} dt.$$

So we have $c^2 < 0$ and this contradicts $c > 0$.

The latter case

First we consider $\Phi = {}^t(\Phi_1, \Phi_2, \Phi_3)$ in Theorem 1.1:

$$\Phi_1 = \left(w - \frac{c^2}{w}\right) c' \frac{dz}{z}, \quad \Phi_2 = i \left(w + \frac{c^2}{w}\right) c' \frac{dz}{z}, \quad \Phi_3 = 2cc' \frac{dz}{z}.$$

For the residue of Φ_3 at $z = 0$ to be real, we see that $c' = \pm 1$. We may choose $c' = 1$.

We shall show that for any $c > 0$ and $a \in \mathbb{R} \setminus \{0, 1\}$, the period condition (P) cannot be satisfied.

A straightforward calculation yields

$$\eta + \frac{3}{2}dw = \left(\frac{w}{z-1} + \frac{w}{z-a}\right) dz. \quad (3.13)$$

Note that the right hand side of this equation is a holomorphic differential on $\overline{M}_2 \setminus \{(\infty, \infty)\}$.

We now consider the following three cases: $a > 1$, $0 < a < 1$, $a < 0$.

(i) The case $a > 1$:

We set

$$\begin{aligned}\ell = & \left\{ (z, w) = \left(t, \sqrt[3]{\frac{(1-t)^2(a-t)^2}{t^2}} \right) \mid 0 \leq t \leq 1 \right\} \\ & \cup \left\{ (z, w) = \left(-t, e^{\frac{2}{3}\pi i} \sqrt[3]{\frac{(1+t)^2(a+t)^2}{t^2}} \right) \mid -1 \leq t \leq 0 \right\}.\end{aligned}$$

From (3.13), we have

$$\oint_{\ell} \eta = -(1 - e^{2\pi i/3}) \int_0^1 \left(\sqrt[3]{\frac{(a-t)^2}{t^2(1-t)}} + \sqrt[3]{\frac{(1-t)^2}{t^2(a-t)}} \right) dt, \quad (3.14)$$

$$\oint_{\ell} g^2 \eta = c^2 (1 - e^{-2\pi i/3}) \int_0^1 \frac{dt}{\sqrt[3]{t(1-t)^2(a-t)^2}}. \quad (3.15)$$

Thus (1.4) is equivalent to

$$-\int_0^1 \left(\sqrt[3]{\frac{(a-t)^2}{t^2(1-t)}} + \sqrt[3]{\frac{(1-t)^2}{t^2(a-t)}} \right) dt = c^2 \int_0^1 \frac{dt}{\sqrt[3]{t(1-t)^2(a-t)^2}}.$$

But this is impossible because the left hand side is a negative real number and the right hand side is a positive real number.

(ii) The case $0 < a < 1$:

We set

$$\begin{aligned} \ell = & \left\{ (z, w) = \left(at, \sqrt[3]{\frac{(at-1)^2(1-t)^2}{t^2}} \right) \mid 0 \leq t \leq 1 \right\} \\ & \cup \left\{ (z, w) = \left(-at, e^{\frac{2}{3}\pi i} \sqrt[3]{\frac{(at+1)^2(1+t)^2}{t^2}} \right) \mid -1 \leq t \leq 0 \right\}. \end{aligned}$$

From (3.13), we have

$$\oint_{\ell} \eta = -(1 - e^{2\pi i/3}) \int_0^1 \left(a \sqrt[3]{\frac{(1-t)^2}{t^2(1-at)}} + \sqrt[3]{\frac{(1-at)^2}{t^2(1-t)}} \right) dt, \quad (3.16)$$

$$\int_{\ell} g^2 \eta = c^2 (1 - e^{-2\pi i/3}) \int_0^1 \frac{dt}{\sqrt[3]{t(1-t)^2(1-at)^2}}. \quad (3.17)$$

Thus (1.4) is equivalent to

$$-\int_0^1 \left(a \sqrt[3]{\frac{(1-t)^2}{t^2(1-at)}} + \sqrt[3]{\frac{(1-at)^2}{t^2(1-t)}} \right) dt = c^2 \int_0^1 \frac{dt}{\sqrt[3]{t(1-t)^2(1-at)^2}},$$

but again this is impossible by the same reason as in the case (i).

(iii) The case $a < 0$:

We set

$$\begin{aligned} \ell = & \left\{ (z, w) = \left(at, \sqrt[3]{\frac{(1-at)^2(1-t)^2}{t^2}} \right) \mid 0 \leq t \leq 1 \right\} \\ & \cup \left\{ (z, w) = \left(-at, e^{\frac{2}{3}\pi i} \sqrt[3]{\frac{(1+at)^2(1+t)^2}{t^2}} \right) \mid -1 \leq t \leq 0 \right\}, \\ \ell' = & \left\{ (z, w) = \left(t, \sqrt[3]{\frac{(1-t)^2(t-a)^2}{t^2}} \right) \mid 0 \leq t \leq 1 \right\} \\ & \cup \left\{ (z, w) = \left(-t, e^{\frac{2}{3}\pi i} \sqrt[3]{\frac{(1+t)^2(t+a)^2}{t^2}} \right) \mid -1 \leq t \leq 0 \right\}. \end{aligned}$$

From (3.13), we have

$$\oint_{\ell} \eta = -(1 - e^{2\pi i/3}) \int_0^1 \left(a \sqrt[3]{\frac{(1-t)^2}{t^2(1-at)}} + \sqrt[3]{\frac{(1-at)^2}{t^2(1-t)}} \right) dt, \quad (3.18)$$

$$\int_{\ell} g^2 \eta = c^2 (1 - e^{-2\pi i/3}) \int_0^1 \frac{dt}{\sqrt[3]{t(1-t)^2(1-at)^2}}. \quad (3.19)$$

Thus (1.4) is equivalent to

$$- \int_0^1 \left(a \sqrt[3]{\frac{(1-t)^2}{t^2(1-at)}} + \sqrt[3]{\frac{(1-at)^2}{t^2(1-t)}} \right) dt = c^2 \int_0^1 \frac{dt}{\sqrt[3]{t(1-t)^2(1-at)^2}}. \quad (3.20)$$

The right hand side is clearly positive. So now we estimate the left hand side.

$$\begin{aligned} (\text{LHS}) &= -a \int_0^1 \sqrt[3]{\frac{(1-t)^2}{t^2(1-at)}} dt - \int_0^1 \sqrt[3]{\frac{(1-at)^2}{t^2(1-t)}} dt \\ &\leq -a \int_0^1 \sqrt[3]{\frac{(1-t)^2}{t^2}} dt - \int_0^1 \sqrt[3]{\frac{1}{t^2(1-t)}} dt \\ &= -aB\left(\frac{1}{3}, \frac{5}{3}\right) - B\left(\frac{1}{3}, \frac{2}{3}\right) \\ &= -\left(\frac{2}{3}a + 1\right) B\left(\frac{1}{3}, \frac{2}{3}\right), \end{aligned}$$

where $B(x, y)$ is the classical beta function mentioned in §2.2. Hence, if $-\left(\frac{2}{3}a + 1\right) \leq 0$, (3.20) never holds. That is,

$$\text{if } -3/2 \leq a < 0, \text{ (3.20) never holds.} \quad (3.21)$$

On the other hand, we have

$$\oint_{\ell'} \eta = -(1 - e^{2\pi i/3}) \int_0^1 \left(\sqrt[3]{\frac{(t-a)^2}{t^2(1-t)}} - \sqrt[3]{\frac{(1-t)^2}{t^2(t-a)}} \right) dt, \quad (3.22)$$

$$\int_{\ell'} g^2 \eta = c^2 (1 - e^{-2\pi i/3}) \int_0^1 \frac{dt}{\sqrt[3]{t(1-t)^2(t-a)^2}}, \quad (3.23)$$

from (3.13). Thus (1.4) is equivalent to

$$- \int_0^1 \left(\sqrt[3]{\frac{(t-a)^2}{t^2(1-t)}} - \sqrt[3]{\frac{(1-t)^2}{t^2(t-a)}} \right) dt = c^2 \int_0^1 \frac{dt}{\sqrt[3]{t(1-t)^2(t-a)^2}}. \quad (3.24)$$

The right hand side is again positive. So now we estimate the left hand side again.

$$\begin{aligned}
(\text{LHS}) &= - \int_0^1 \sqrt[3]{\frac{(t-a)^2}{t^2(1-t)}} dt + \int_0^1 \sqrt[3]{\frac{(1-t)^2}{t^2(t-a)}} dt \\
&\leq - \int_0^1 \sqrt[3]{\frac{(-a)^2}{t^2(1-t)}} dt + \int_0^1 \sqrt[3]{\frac{(1-t)^2}{t^2(-a)}} dt \\
&= -(-a)^{2/3} B\left(\frac{1}{3}, \frac{2}{3}\right) + (-a)^{-1/3} B\left(\frac{1}{3}, \frac{5}{3}\right) \\
&= (-a)^{-1/3} \left(a + \frac{2}{3}\right) B\left(\frac{1}{3}, \frac{2}{3}\right).
\end{aligned}$$

Hence, if $a + \frac{2}{3} \leq 0$, (3.24) never holds. That is,

$$\text{if } a \leq -2/3, (3.24) \text{ never holds.} \quad (3.25)$$

Combining (3.21) and (3.25), the period condition cannot be solved.

Main Theorem 2 is an immediate consequence by the above arguments.

4 Remaining problems

In this section we introduce remaining problems related to this work.

4.1 The case that γ odd and greater than 1

For the case that the genus γ is odd and greater than 1, a complete minimal surface of finite total curvature $f : M = \overline{M}_\gamma \setminus \{p_1, p_2\} \rightarrow \mathbb{R}^3$ which satisfies equality in (1.9) is yet to be found. However, Matthias Weber [We] has constructed the following examples numerically.

Example 4.1 (Weber). Let γ be a positive integer. Define

$$F_1(z; a_1, a_3, \dots, a_{2\gamma-1}) = \prod_{i=1}^{\gamma} (z - a_{2i-1}), \quad F_2(z; a_2, a_4, \dots, a_{2\gamma}) = \prod_{i=1}^{\gamma} (z - a_{2i}),$$

where $1 = a_1 < a_2 < \dots < a_{2\gamma}$ are constants to be determined. Define a compact Riemann surface \overline{M}_γ of genus γ by

$$\overline{M}_\gamma = \left\{ (z, w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^2 = z \frac{F_1(z; a_1, a_3, \dots, a_{2\gamma-1})}{F_2(z; a_2, a_4, \dots, a_{2\gamma})} \right\}.$$

We set

$$M = \overline{M}_\gamma \setminus \{(0, 0), (\infty, \infty)\}, \quad g = c \frac{w}{z+1} \quad (c > 0), \quad \eta = \frac{(z+1)^2}{zw} dz.$$

Then there exist constants $c, a_2, a_3, \dots, a_{2\gamma}$ such that (P) holds. (See Figure 4.1.)

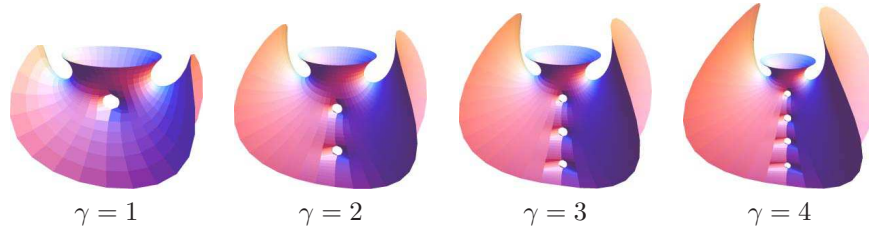


Figure 4.1: Minimal surfaces of genus γ with two ends which satisfy $\deg(g) = \gamma + 2$.

For $\gamma = 1$, we can prove the existence of the surface rigorously. However, for other cases, since the surface does not have enough symmetry, the rigorous proof of the existence still remains an open problem.

4.2 Existence of non-orientable minimal surfaces

Our work is devoted to minimal surfaces satisfying $\deg(g) = \gamma + 2$. On the other hand, it is important to consider the existence of non-orientable minimal surfaces with $\deg(g) = \gamma + 3$. Now, we review non-orientable minimal surfaces in \mathbb{R}^3 .

Let $f' : M' \rightarrow \mathbb{R}^3$ be a minimal immersion of a non-orientable surface into \mathbb{R}^3 . Then the oriented two sheeted covering space M of M' naturally inherits a Riemann surface structure and we have a canonical projection $\pi : M \rightarrow M'$. We can also define a map $I : M \rightarrow M$ such that $\pi \circ I = \pi$, which is an antiholomorphic involution on M without fixed points. Here M' can be identified with $M/\langle I \rangle$. In this way, if $f : M \rightarrow \mathbb{R}^3$ is a conformal minimal surface and there is an antiholomorphic involution $I : M \rightarrow M$ without fixed points so that $f \circ I = f$, then we can define a non-orientable minimal surface $f' : M' = M/\langle I \rangle \rightarrow \mathbb{R}^3$. Conversely, every non-orientable minimal surface is obtained in this procedure.

Suppose that $f' : M' = M/\langle I \rangle \rightarrow \mathbb{R}^3$ is complete and of finite total curvature. Then, we can apply Theorems 1.4 and 1.5 to the conformal minimal immersion $f : M \rightarrow \mathbb{R}^3$. Furthermore, we have a stronger restriction on the topology of M' or M . In fact, Meeks [Me] showed that the Euler characteristic $\chi(\overline{M}_\gamma)$ and $2\deg(g)$ are congruent modulo 4, where g is the Gauss map of f . By these facts, we can observe that for every complete non-orientable minimal surface of finite total curvature, $\deg(g) \geq \gamma + 3$ holds.

For $\gamma = 0$ and $\gamma = 1$, Meeks' Möbius strip [Me] and López' Klein bottle [L2] satisfy $\deg(g) = \gamma + 3$, respectively. But, for $\gamma \geq 2$, no examples with $\deg(g) = \gamma + 3$ are known. So, it is interesting to give a minimal surface satisfying $\deg(g) = \gamma + 3$ with an antiholomorphic involution without fixed points. This problem appeared in [LM] and [Ma].

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